tkz-elements 2.20c

AlterMundus

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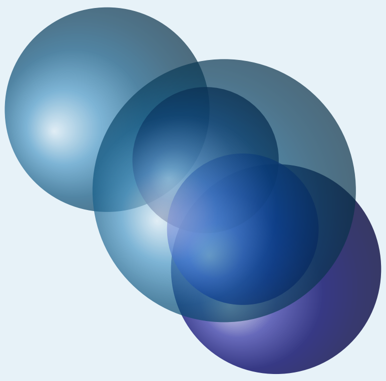
Euclidean Geometry



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http://altermundus.fr

tkz-elements

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AlterMundus

☞ This document compiles some notes about **`tkz-elements`**, the initial version of a `Lua` library designed to perform all the necessary calculations for defining objects in Euclidean geometry figures. Your document must be compiled using LuaLaTeX.
With `tkz-elements`, definitions and calculations are exclusively conducted using `Lua`.
The primary programming approach offered is oriented towards `object programming`, utilizing object classes such as point, line, triangle, circle, and ellipse. Currently, after the calculations are completed, `tkz-euclide` or `TikZ` is used for drawing purposes.
I discovered Lua and object-oriented programming while developing this package, so it's highly likely that I've made a few mistakes. If you'd like to contribute to the development of this package or provide advice on how to proceed, please contact me via email.
Please note: English is not my native language, so there may be some errors."

☞ Acknowledgements : I received much valuable advices, remarks, corrections from `Nicolas Kisselhoff`, `David Carlisle`, `Roberto Giacomelli` and `Qrrbrbirlbel`.
Special thanks to `Wolfgang Büchel` for his invaluable contribution in correcting the examples.

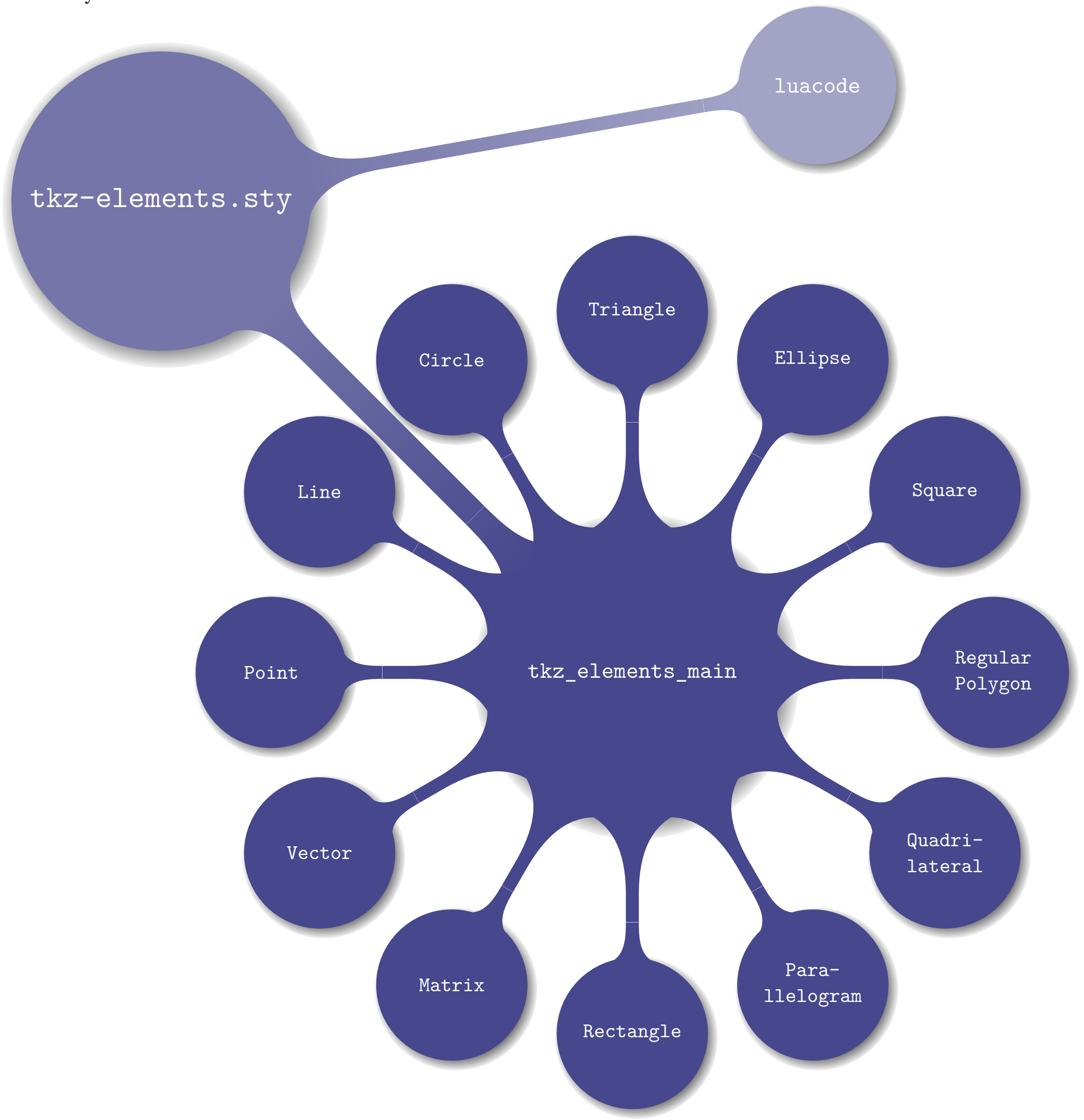
☞ I would also like to extend my gratitude to `Eric Weisstein`, creator of MathWorld.

☞ You can find some examples on my site: altermundus.fr. under construction!

Contents

1 Structure

`tkz-elements.sty` loads the `luacode` package to create the **`tkzelements`** environment, which is based on the **`luacode`** environment.
Within the **`tkzelements`** environment, the scale is initialized to 1, and then all values in various tables are cleared.
The package defines two macros `\tkzGetNodes` and `\tkzUseLua`.
Additionally, the package loads the file `tkz_elements_main.lua`. This file initializes all the tables that will be used by the modules in which the classes are defined.



The current classes are (some are still inactive):

- active : `point` (z) ; `line` (L) ; `circle` (C) ; `triangle` (T) ; `ellipse` (E) ; `quadrilateral` (Q) ; `square` (S) ; `rectangle` (R) ; `parallelogram` (P) ; `regular_polygon` (RP); `vector` (V).
- inactive : matrix (M) ; vector (V).

If name is name of a class, you can find its definition in the file `tkz_elements_name.lua`.

2 Why tkz-elements?

2.1 Calculation accuracy

2.1.1 Calculation accuracy in TikZ

With Ti*k*Z, the expression `veclen(x,y)` calculates the expression $\sqrt{x^2+y^2}$. This calculation is achieved through a polynomial approximation, drawing inspiration from the ideas of `Rouben Rostamian`.

```
    pgfmathparse{veclen(65,72)} \pgfmathresult
```

☞ $\sqrt{65^2+72^2} \approx 96.9884$ 💣.

2.1.2 Calculation accuracy in Lua

A `luaveclen` macro can be defined as follows:

```
\def\luaveclen#1#2{\directlua{tex.print(string.format(
'\percentchar.5f',math.sqrt((#1)*(#1)+(#2)*(#2))))}}
```

and

```
\luaveclen{65}{72}
```

gives
☞ $\sqrt{65^2+72^2} = 97$ ‼
The error, though insignificant when it comes to the placement of an object on a page by a hundredth of a point, becomes problematic for the results of mathematical demonstrations. Moreover, these inaccuracies can accumulate and lead to erroneous constructions.

To address this lack of precision, I initially introduced the `fp`, followed by the package `xfp`. More recently, with the emergence of LuaLaTeX, I incorporated a **Lua** option aimed at performing calculations with **Lua**.
This was the primary motivation behind creating the package, with the secondary goal being the introduction of object-oriented programming (OOP) and simplifying programming with Lua. The concept of OOP persuaded me to explore its various possibilities further.
At that time, I had received some Lua programming examples from `Nicolas Kisselhoff`, but I struggled to understand the code initially, so I dedicated time to studying Lua patiently. Eventually, I was able to develop **tkz-elements**, incorporating many of his ideas that I adapted for the package.

2.1.3 Using objects

Subsequently, I came across an article by `Roberto Giacomelli`[1] on object-oriented programming using **Lua** and Ti*k*Z tools. This served as my second source of inspiration. Not only did this approach enable programming to be executed step-by-step, but the introduction of objects facilitated a direct link between the code and geometry. As a result, the code became more readable, explicit, and better structured.

2.1.4 Example: Apollonius circle

> Problem: The objective is to identify an inner tangent circle to the three exinscribed circles of a triangle.

For additional details, refer to MathWorld for more details.

1 Grafica ad oggetti con LuaTEX

This example served as my reference for testing the `tkz-euclide` package. Initially, with my first methods and the tools available to me, the results lacked precision. However, with tkz-elements, I now have access to more powerful and precise tools that are also easier to use.
The fundamental principles of figure construction with **`tkz-euclide`** remain intact: definitions, calculations, tracings, labels, as well as the step-by-step programming, mirroring the process of construction with a ruler and compass.
This version utilizes the simplest construction method made possible by Lua.

```
\begin{tkzelements}
  scale           = .4
  z.A             = point: new (0,0)
  z.B             = point: new (6,0)
  z.C             = point: new (0.8,4)
  T.ABC           = triangle : new ( z.A,z.B,z.C )
  z.N             = T.ABC.eulercenter
  z.S             = T.ABC.spiekercenter
  T.feuerbach     = T.ABC : feuerbach ()
  z.Ea,z.Eb,z.Ec  = get_points ( T.feuerbach )
  T.excentral     = T.ABC : excentral ()
  z.Ja,z.Jb,z.Jc  = get_points ( T.excentral )
  C.JaEa          = circle: new (z.Ja,z.Ea)
  C.ortho         = circle: radius (z.S,math.sqrt(C.JaEa: power(z.S)))
  z.a             = C.ortho.through
  C.euler         = T.ABC: euler_circle ()
  C.apo           = C.ortho : inversion (C.euler)
  z.O             = C.apo.center
  z.xa,z.xb,z.xc  = C.ortho : inversion (z.Ea,z.Eb,z.Ec)
\end{tkzelements}
```

The creation of an object encapsulates its attributes (its characteristics) and methods (i.e. the actions that are specific to it). Subsequently, it is assigned a reference (a name) which is linked to the object using a table. This table functions as an associative array that links the reference, called a `key`, to a `value`, in this case, the object. Further elaboration on these notions will be provided later.
For instance, let `T` be a table associating the object `triangle` with the key `ABC`. `T.ABC` is also a table, and its elements are accessed using keys that correspond to attributes of the triangle. These attributes have been defined within the package.

```
 z.N = T.ABC.eulercenter
```

`N` is the name of the point, `eulercenter` is an attribute of the triangle. [2]

```
 T.excentral     = T.ABC : excentral ()
```

In this context, `excentral` is a method associated with the `T.ABC` object. It defines the triangle formed by the centers of the exinscribed circles.
Of particular importance are two lines of code. The first one below demonstrates that the exceptional precision provided by Lua allows for the definition of a radius through a complex calculation. The radius of the radical circle is determined by $\sqrt{\Pi(S,\mathcal{C}(Ja,Ea))}$ (square root of the power of point S with respect to the exinscribed circle with center `Ja` passing through `Ea`).

2 The center of the Euler circle, or center of the nine-point circle, is a characteristic of every triangle.

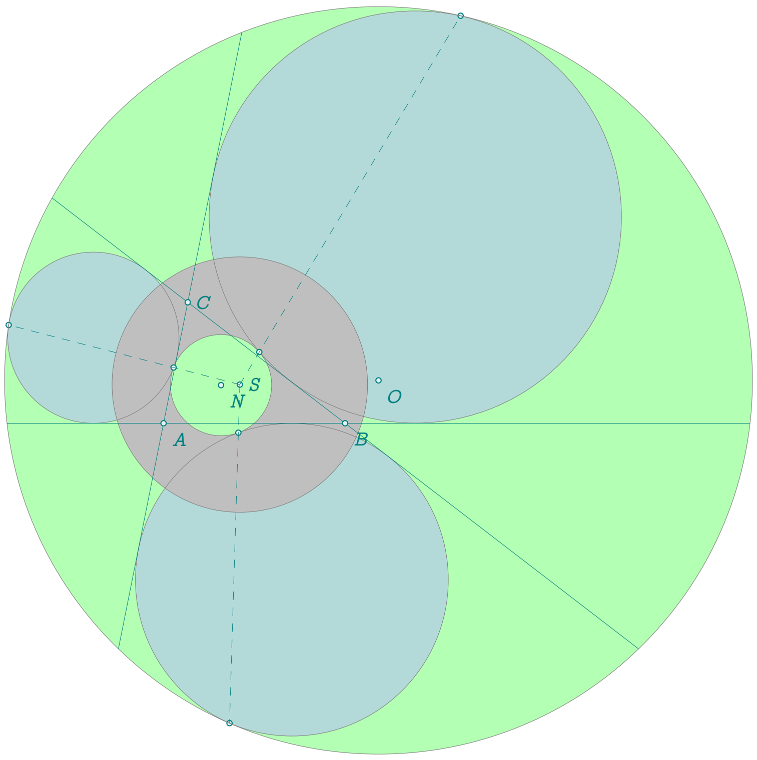
```
C.ortho  = circle: radius (z.S,math.sqrt(C.JaEa: power(z.S)))
```

Lastly, it's worth noting that the inversion of the Euler circle with respect to the radical circle yields the Apollonius circle[3]. This transformation requires an object as a parameter, which is recognized by its type (all objects are typed in the package), and the method determines which algorithm to use according to this type.

```
C.apo   = C.ortho : inversion (C.euler)
```

Now that all the points have been defined, it's time to start drawing the paths. To accomplish this, nodes need to be created. This is the role of the macro . Refer to 6.1.1
The subsequent section exclusively deals with drawings, and is managed by `tkz-euclide`.

```
\begin{tikzpicture}
   \tkzGetNodes
   \tkzFillCircles[green!30](O,xa)
   \tkzFillCircles[teal!30](Ja,Ea Jb,Eb Jc,Ec)
   \tkzFillCircles[lightgray](S,a)
   \tkzFillCircles[green!30](N,Ea)
   \tkzDrawPoints(xa,xb,xc)
   \tkzDrawCircles(Ja,Ea Jb,Eb Jc,Ec S,a O,xa N,Ea)
   \tkzClipCircle(O,xa)
   \tkzDrawLines[add=3 and 3](A,B A,C B,C)
   \tkzDrawPoints(O,A,B,C,S,Ea,Eb,Ec,N)
   \tkzDrawSegments[dashed](S,xa S,xb S,xc)
   \tkzLabelPoints(O,N,A,B)
   \tkzLabelPoints[right](S,C)
\end{tikzpicture}
```



3 The nine-point circle, or Euler circle, is externally tangent to the three circles. The points of tangency form Feuerbach's triangle.

3 Presentation

3.1 With Lua

The primary function of tkz-elements is to calculate dimensions and define points, which is achieved using Lua. You can view tkz-elements as a kernel that is utilized either by tkz-euclide or by TikZ, Definitions and calculations take place within the environment **tkzelements**, which is based on **luacode**.

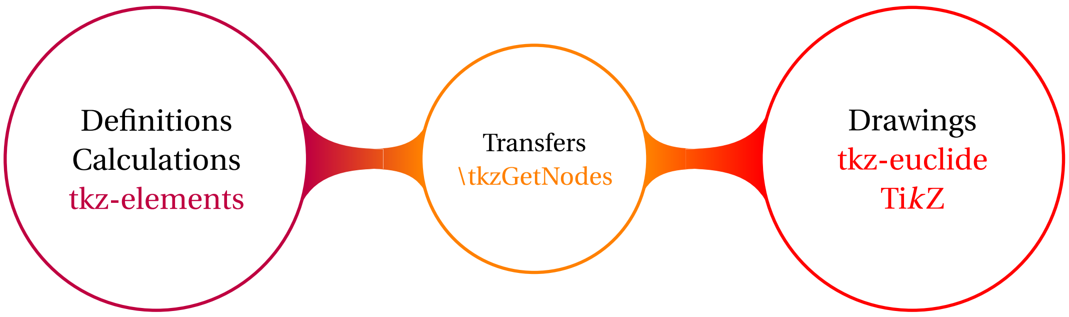
The key points are:

- The source file must be ☞ UTF8 encoded.
- Compilation is done with ☞ LuaLaTeX.
- You need to load `TikZ` or `tkz-euclide` and `tkz-elements`.
- Definitions and calculations are performed in an (orthonormal) Cartesian coordinate system, using Luawithin the `tkzelements` environment.

On the right, you can see the minimum template.
The code is divided into two parts, represented by two environments **tkzelements** and **tikzpicture**. In the first environment, you place your Lua code, while in the second, you use tkz-euclide commands.

```
% !TEX TS-program = lualatex
% Created by Alain Matthes
\documentclass{standalone}
\usepackage{tkz-euclide}
% or simply TikZ
\usepackage{tkz-elements}
begin{document}

\begin{tkzelements}
   scale = 1
% definition of some points
z.A = point : new (   ,   )
z.B = point : new (   ,   )

 ...code...
\end{tkzelements}

\begin{tikzpicture}
% point transfer to Nodes
\tkzGetNodes

\end{tikzpicture}
\end{document}
```

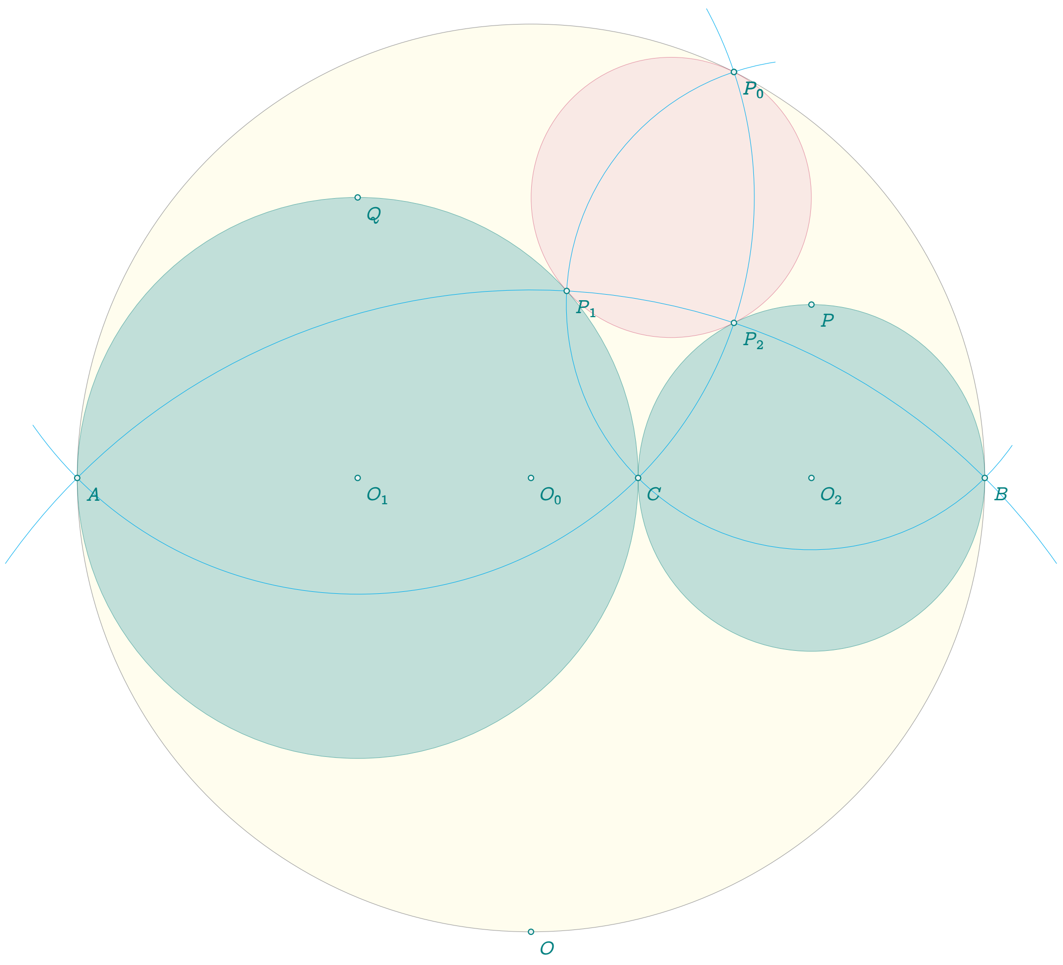
3.2 The main process



After obtaining all the necessary points for the drawing, they must be transformed into **nodes** so that `TikZ` or `tkz-euclide` can render the figure. This is accomplished using the macro `\tkzGetNodes`. This macro iterates through all the elements of the table z using the key (which is essentially the name of the point) and retrieves the associated values, namely the coordinates of the point (node).

3.3 Complete example: Pappus circle

3.3.1 The figure



3.3.2 The code

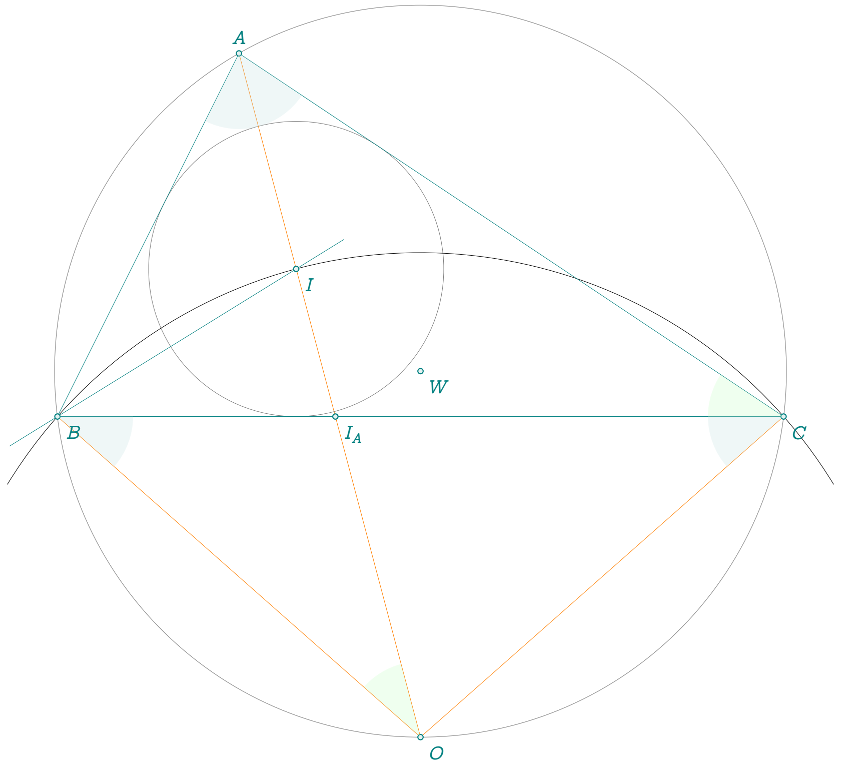
```
% !TEX TS-program = lualatex
\documentclass{article}
\usepackage{tkz-euclide}
\usepackage{tkz-elements}
\begin{document}

\begin{tkzelements}
z.A     = point: new (0 , 0)
z.B     = point: new (10 , 0)         --  creation of two fixed points $A$ and $B$
L.AB    = line:  new ( z.A, z.B)
z.C     = L.AB:  gold_ratio ()        --  use of a method linked to "line"
z.O_0   = line:  new ( z.A, z.B).mid  -- midpoint of segment with an attribute of "line"
z.O_1   = line:  new ( z.A, z.C).mid --  objects are not stored and cannot be reused.
z.O_2   = line:  new ( z.C, z.B).mid
C.AB    = circle: new ( z.O_0, z.B)  --  new object "circle" stored and reused
C.AC    = circle: new ( z.O_1, z.C)
C.CB    = circle: new ( z.O_2, z.B)
z.P     = C.CB.north                 --  "north" atrributes of a circle
```

```
z.Q     = C.AC.north
z.O     = C.AB.south
z.c     = z.C : north (2)           --   "north" method of a point (needs a parameter)
C.PC    = circle: new ( z.P, z.C)
C.QA    = circle: new ( z.Q, z.A)
z.P_0   = intersection (C.PC,C.AB)   --  search for intersections of two circles.
z.P_1   = intersection (C.PC,C.AC)  --   idem
_,z.P_2 = intersection (C.QA,C.CB)   --  idem
z.O_3   = triangle: new ( z.P_0, z.P_1, z.P_2).circumcenter -- circumcenter attribute of "triangle"
\end{tkzelements}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawCircle[black,fill=yellow!20,opacity=.4](O_0,B)
  \tkzDrawCircles[teal,fill=teal!40,opacity=.6](O_1,C O_2,B)
  \tkzDrawCircle[purple,fill=purple!20,opacity=.4](O_3,P_0)
  \tkzDrawArc[cyan,delta=10](Q,A)(P_0)
  \tkzDrawArc[cyan,delta=10](P,P_0)(B)
  \tkzDrawArc[cyan,delta=10](O,B)(A)
  \tkzDrawPoints(A,B,C,O_0,O_1,O_2,P,Q,P_0,P_0,P_1,P_2,O)
  \tkzLabelPoints(A,B,C,O_0,O_1,O_2,P,Q,P_0,P_0,P_1,P_2,O)
\end{tikzpicture}
\end{document}
```

3.4 Another example with comments: South Pole

Here's another example with comments

```
% !TEX TS-program = lualatex
\documentclass{standalone}
\usepackage{tkz-euclide,tkz-elements}
\begin{document}
\begin{tkzelements}
   z.A      = point: new (2 , 4)           -- we create environment tkzelements
   z.B      = point: new (0 , 0)           -- three fixed points are used
   z.C      = point: new (8 , 0)
   T.ABC    = triangle: new (z.A,z.B,z.C) -- we create a new triangle object
   C.ins    = T.ABC: in_circle ()          -- we get the incircle of this triangle
   z.I      = C.ins.center                 -- center is an attribute of the circle
   z.T      = C.ins.through                -- through is also an attribute
   -- z.I,z.T  = get_points (C.ins)        -- get_points is a shortcut
   C.cir    = T.ABC : circum_circle ()     -- we get the  circumscribed circle
   z.W      = C.cir.center                 -- we get the center of this circle
   z.O      = C.cir.south                  -- now we get the south pole of this circle
   L.AO     = line: new (z.A,z.O)          -- we create an object "line"
   L.BC     = T.ABC.bc                     -- we get the line (BC)
   z.I_A    = intersection (L.AO,L.BC)     --  we search the intersection of the last lines
\end{tkzelements}
```



Here's the tikzpicture environment to obtain the drawing:

```
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawCircles(W,A I,T)
\tkzDrawArc(O,C)(B)
\tkzDrawPolygon(A,B,C)
\tkzDrawSegments[new](A,O B,O C,O)
\tkzDrawLine(B,I)
\tkzDrawPoints(A,B,C,I,I_A,W,O)
\tkzFillAngles[green!20,opacity=.3](A,O,B A,C,B)
\tkzFillAngles[teal!20,opacity=.3](O,B,C B,C,O B,A,O O,A,C)
\tkzLabelPoints(I,I_A,W,B,C,O)
\tkzLabelPoints[above](A)
\end{tikzpicture}
```

4 Writing Convention

4.1 Miscellaneous

- Numerical variable: the writing conventions for real numbers are the same as for Lua.
- Complex numbers: Similar to real numbers, but to define them, you must write `za = point (1,2)`. Mathematically, this corresponds to 1+2i, which you can find with `tex.print(tostring(za))`.(Refer 22.3)
- Boolean: you can write `bool = true` or `bool = false` then with Lua you can use the code :

  ```
  if bool == ... then ... else ... end
  ```

  and outside the environment **`tkzelements`** you can use the macro

  ```
  \ifthenelse{\equal{\tkzUseLua{bool}}{true}}{ ... }{ ... }
  ```

  after loading the `ifthen` package.
- String: if st = ”Euler’s formula” then

  ```
  \tkzUseLua{st} gives Euler's formula
  ```

4.2 Assigning a Name to a Point

At present, the only obligation is to store the points in the table `z` [4] if you intend to use them in Ti*k*Z or `tkz-euclide`. f a point will not be used, you can designate it as you wish while adhering to Lua conventions.
Points within the **`tkzelements`** environment must follow a convention in the form `z.name`, where `name` represents the name of the corresponding **`node`**.
As for the conventions for designating `name` you must adhere to Lua conventions in particular cases.

1. The use of prime can be problematic. If the point name contains more than one symbol and ends with `p` then when passing into `TikZ` or `tkz-euclide`, the letters `p` will be replaced by `'` using the macro **`\tkzGetNodes`**;

2. Alternatively, for a more explicit code, suppose you want to designate a point as ”euler”. You could, for example, write `euler = ...`, and at the end of the code for the transfer, `z.E = euler`. It is also possible to use a temporary name `euler` and to replace it in Ti*k*Z. Either at the time of placing the labels, or for example by using `pgfnodealias{E}{euler}`. This possibility also applies in other cases: prime, double prime, etc.

Here are some different ways of naming a point:

- `z.A = point : new (1,2)`
- `z.Bp = point : new (3,4)` –> this gives `B'` in the **`tikzpicture`**
- `z.H_a = T.ABC : altitude ()` –> this gives `H_a` in the **`tikzpicture`** code and H_a in the display.

4 To place the point M in the table, simply write `z.M` = …or `z["M"]` = …

4.3 Assigning a Name to Other Objects

You have the flexibility to assign names to objects other than points. However, it's advisable to adhere to certain conventions to enhance code readability. For my examples, I've chosen the following conventions: first of all, I store the objects in tables: `L` for lines and segments, `C` for circles, `T` for triangles, `E` for ellipses.

- For lines, I use the names of the two points they pass through. For example, if a line passes through points A and B, I name the line `L.AB`.
- Circles are stored in table named `C`. For example, I name `C.AB` the circle of center A passing through B. Other names like C.euler or C.external are also acceptable.
- Triangles are stored in table named `T`. For example, I name `T.ABC` the triangle whose vertices are A, B and C. However, names like `T.feuerbach` are also acceptable.
- Ellipses are stored in table named `E`. For ellipses, I name `E.ABC` the ellipse with center A through vertex B and covertex C.

Adhering to these conventions can help improve the readability of the code.

4.4 Writing conventions for attributes, methods.

You must use the conventions of Lua, so

- To obtain an , for all objects, the convention is identical: `object.attribute`. For example, for the point A we access its abscissa with `z.A.re` and its ordinate with `z.A.im`; as for its type we obtain it with `z.A.type`. To get the south pole of the circle `C.OA` you need to write: `C.OA.south`.
- To use a method such as obtaining the incircle of a triangle ABC, just write

  `C.incircle = T.ABC : in_circle ().`
- Some methods need a parameter. For example, to know the distance between a point C to the line (A, B) we will write

  `d = L.AB : distance (z.C).`
- Use the to store a result you don't want to use. If you only need the second point of an intersection between a line and a circle, you would write

  `_,z.J = intersection (L.AB , C.OC).`

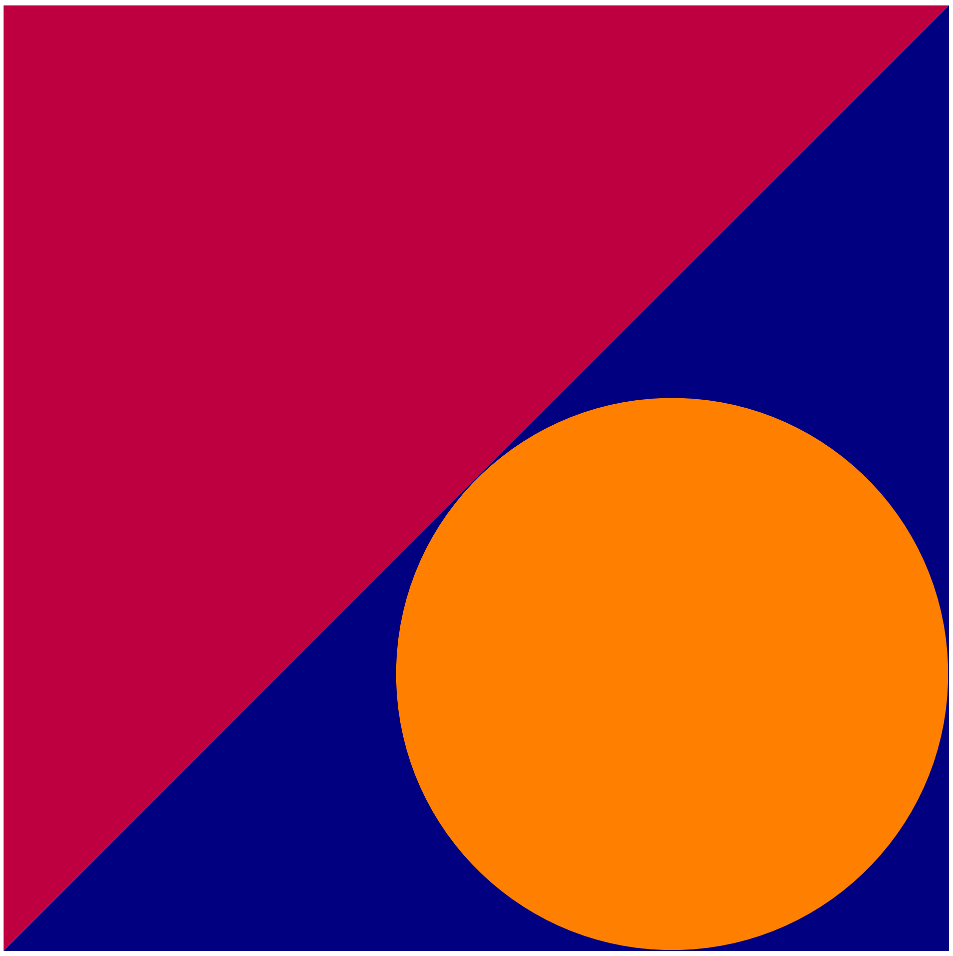
5 Work organization

Here's a sample organization.
The line `% !TEX TS-program = lualatex` ensures that you compile with LuaLaTeX. The `standalone` class is useful, as all you need to do here is create a figure.
The package `ifthen` is useful if you need to use some Boolean.
The macro `\LuaCodeDebugOn` allows you to try and find errors in Lua code.
While it's possible to leave the Lua code in the **`tkzelements`** environment, externalizing this code has its advantages.
The first advantage is that, if you use a good editor, you have a better presentation of the code. Styles differ between `Lua` and LaTeX, making the code clearer. This is how I proceeded, then reintegrated the code into the main code.
Another advantage is that you don't have to incorrectly comment the code. For Lua code, you comment lines with `--` (double minus sign), whereas for LaTeX, you comment with `%`.
A third advantage is that the code can be reused.

```
% !TEX TS-program = lualatex
% Created by Alain Matthes on 2024-01-09.

\documentclass[margin = 12pt]{standalone}
\usepackage{tkz-euclide}
\usepackage{tkz-elements,ifthen}

\begin{document}
\LuaCodeDebugOn
\begin{tkzelements}
 scale = 1.25
 dofile ("sangaku.lua")
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(I,F)
   \tkzFillPolygon[color = purple](A,C,D)%
   \tkzFillPolygon[color = blue!50!black](A,B,C)%
   \tkzFillCircle[color = orange](I,F)%
\end{tikzpicture}
\end{document}
```

And here is the code for the Lua part: the file `ex_sangaku.lua`

```
z.A         = point : new ( 0,0 )
z.B         = point : new ( 8,0 )
L.AB        = line : new ( z.A , z.B )
S           = L.AB : square ()
_,_,z.C,z.D = get_points (S)
z.F         = S.ac : projection (z.B)
L.BF        = line : new (z.B,z.F)
T.ABC       = triangle : new ( z.A , z.B , z.C )
L.bi        = T.ABC : bisector (2)
z.c         = L.bi.pb
L.Cc        = line : new (z.C,z.c)
z.I         = intersection (L.Cc,L.BF)
```



5.1 Scale problem

If necessary, it's better to perform scaling in the `Lua` section. This approach tends to be more accurate. However, there is a caveat to be aware of. I've made it a point to avoid using numerical values in my codes whenever possible. Generally, these values only appear in the definition of fixed points. If the `scale` option is used, scaling is applied when points are created. Let's imagine you want to organize your code as follows:

```
scale = 1.5
xB = 8
z.B         = point : new ( xB,0 )
```

Scaling would then be ineffective, as the numerical values are not modified, only the point coordinates. To account for scaling, use the function `value (v)` .

```
scale = 1.5
xB = value (8)
z.B         = point : new ( xB,0 )
```

5.2 Code presentation

The key point is that, unlike LaTeX or TeX, you can insert spaces absolutely anywhere.

6 Transfers

6.1 Fom Lua to tkz-euclide or TikZ

In this section, we'll explore how to transfer points, Booleans, and numerical values.

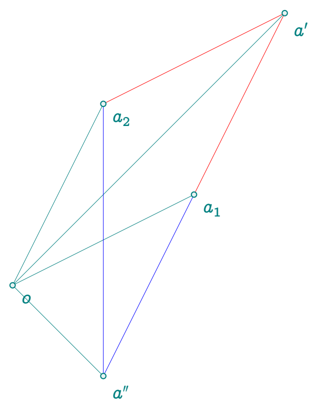
6.1.1 Points transfer

We utilize an environment **tkzelements** outside an **tikzpicture** environment which allows us to perform all the necessary calculations. Then, we execute the macro which transforms the affixes of the table z into **Nodes**. Finally, we proceed with the drawing.
At present, the drawing program is either Ti*k*Z or `tkz-euclide`. However, you have the option to use another package for plotting. To do so, you'll need to create a macro similar to `\tkzGetNodes`. Of course, this package must be capable of storing points like Ti*k*Z or `tkz-euclide`.

```
\def\tkzGetNodes{\directlua{%
   for K,V in pairs(z) do
      local n,sd,ft
      n = string.len(K)
      if n >1 then
      _,_,ft, sd = string.find( K , "(.+)(.)" )
     if sd == "p" then   K=ft.."'" end
     _,_,xft, xsd = string.find( ft , "(.+)(.)" )
     if xsd == "p" then  K=xft.."'".."'" end
       end
  tex.print("\\coordinate ("..K..") at ("..V.re..","..V.im..") ;\\\\")
end}
}
```

See the section In-depth Study 22 for an explanation of the previous code.
The environment **tkzelements** allows to use the underscore _ and the macro `\tkzGetNodes` allows to obtain names of nodes containing **prime** or **double prime**. (Refer to the next example)

```
\begin{tkzelements}
   scale = 1.2
   z.o   = point: new (0,0)
   z.a_1 = point: new (2,1)
   z.a_2 = point: new (1,2)
   z.ap  = z.a_1 + z.a_2
   z.app = z.a_1 - z.a_2
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawSegments(o,a_1 o,a_2 o,a' o,a'')
   \tkzDrawSegments[red](a_1,a' a_2,a')
   \tkzDrawSegments[blue](a_1,a'' a_2,a'')
   \tkzDrawPoints(a_1,a_2,a',o,a'')
   \tkzLabelPoints(o,a_1,a_2,a',a'')
\end{tikzpicture}
```

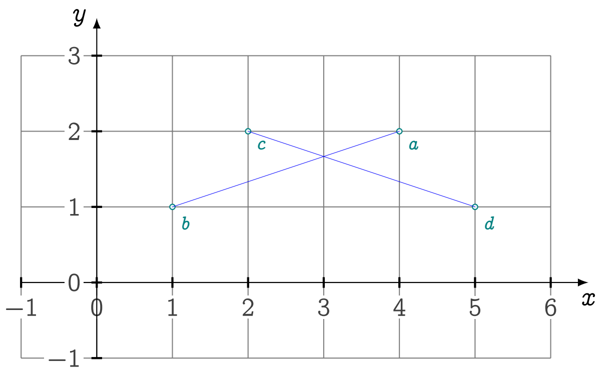


6.1.2 Other transfers

Sometimes it's useful to transfer angle, length measurements or boolean. For this purpose, I have created the macro (refer to 20.6) `tkzUseLua(value)`

```
\begin{tkzelements}
   z.b = point:  new (1,1)
   z.a = point:  new (4,2)
   z.c = point:  new (2,2)
   z.d = point:  new (5,1)
   L.ab = line : new (z.a,z.b)
   L.cd = line : new (z.c,z.d)
    det = (z.b-z.a)^(z.d-z.c)
    if det == 0 then bool = true
      else bool = false
    end
    x = intersection (L.ab,L.cd)
\end{tkzelements}

The intersection of the two lines lies at
    a point whose affix is:\tkzUseLua{x}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzInit[xmin =-1,ymin=-1,xmax=6,ymax=3]
  \tkzGrid\tkzAxeX\tkzAxeY
   \tkzDrawPoints(a,...,d)
   \ifthenelse{\equal{\tkzUseLua{bool}}{true}}{
   \tkzDrawSegments[red](a,b c,d)}{%
   \tkzDrawSegments[blue](a,b c,d)}
    \tkzLabelPoints(a,...,d)
\end{tikzpicture}
```

The intersection of the two lines lies at a point whose affix is: 3+1.67i



7 Class and object

7.1 Class

Object-oriented programming (OOP) is a programming model based on the concept of objects. An object can be defined as a data table that has unique attributes and methods (operations) that define its behavior.

A class is essentially a user-defined data type. It describes the contents of the objects that belong to it. A class serves as a blueprint for creating objects, providing initial values for attributes and implementations of methods[5] cthat are common to all objects of a certain kind.

7.2 Object

An Object is an instance of a class. Each object contains attributes and methods. Attributes are information or object characteristics of the object stored in the data table (called fields), while methods define the object's behavior.

All objects in the package are typed. The object types currently defined and used are: **point**, **line**, **circle**, **triangle**, **ellipse**, **quadrilateral**, **square**, **rectangle**, **parallelogram** and **regular_polygon**.
These objects can be created directly using the method new by giving points, with the exception of the classpoint class which requires a pair of reals, and classregular_polygon which needs two points and an integer.
Objects can also be obtained by applying methods to other objects. For example, T.ABC : circum_circle () creates an object **circle**. Some object attributes are also objects themselves, such as T.ABC.bc which creates the **line** object, representing a straight line passing through the last two points defining the triangle.

7.2.1 Attributes

Attributes are accessed using the classic method, so T.pc retrieves the third point of the triangle and C.OH.center retrieves the center of the circle. Additionally, I've added a get_points function that returns the points of an object. This function applies to straight lines (pa and pc), triangles (pa, pb and pc) and circles (center and through).

Example: z.O,z.T = get_points (C) retrieves the center and a point of the circle.

7.2.2 Methods

A method is an operation (function or procedure) associated (linked) with an object.
Example: The point object is used to vertically determine a new point object located at a certain distance from it (here 2). Then it is possible to rotate objects around it.

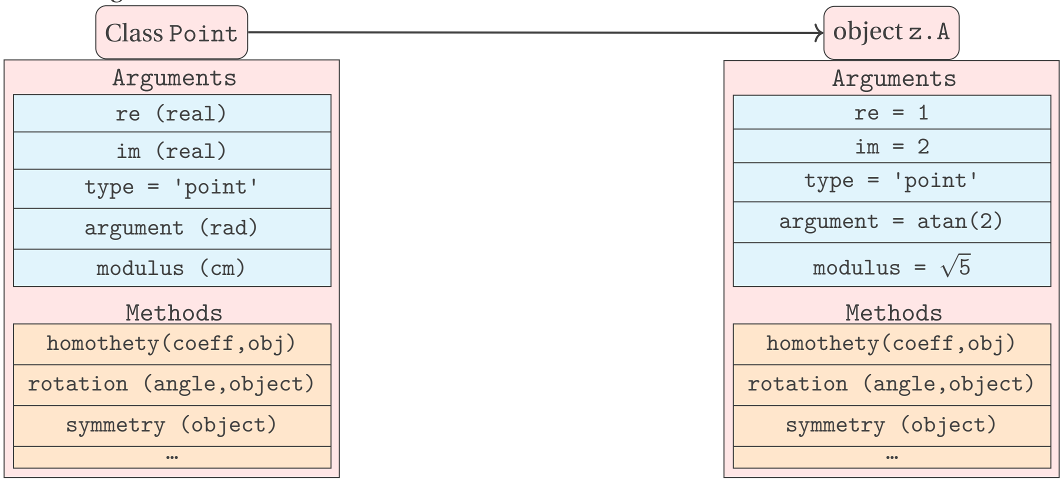
```
\begin{tkzelements}
   z.A = point (1,0)
   z.B = z.A : north (2)
   z.C = z.A : rotation (math.pi/3,z.B)
   tex.print(tostring(z.C))
\end{tkzelements}
```

The coordinates of C are: -0.73205080756888 and 1.0

5 action which an object is able to perform.

8 Class point

The foundation of the entire framework is the point class. This class is hybrid in the sense that it deals with both points in a plane and complex numbers. The principle is as follows: the plane is equipped with an orthonormal basis, which allows us to determine the position of a point using its abscissa and ordinate coordinate. Similarly, any complex number can be viewed simply as a pair of real numbers (its real part and its imaginary part). We can then designate the plane as the complex plane, and the complex number $x+iy$ is represented by the point of the plane with coordinates (x,y). Thus the point A will have coordinates stored in the object $z.A$. Coordinates are attributes of the "point" object, along with type, argument, and modulus.
The creation of a point is done using the following method, but there are other possibilities. If a scaling factor has been given, the method takes it into account.



8.1 Attributes of a point

Creation `z.A = point: new (1,2)`

The point A has coordinates $x=1$ and $y=2$. If you use the notation `z.A`, then A will be referenced as a node in Ti*k*Z or in `tkz-euclide`.
This is the creation of a fixed point with coordinates 1 and 2 and which is named A. The notation `z.A` indicates that the coordinates will be stored in a table denoted as `z` (reference to the notation of the affixes of the complex numbers) that A is the name of the point and the key allowing access to the values.

Table 1: Point attributes.

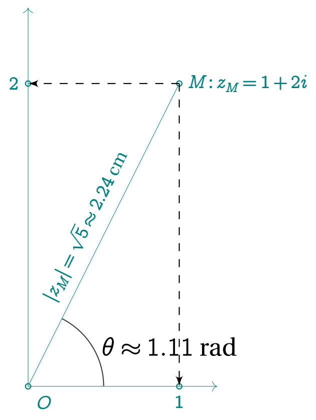
Attributes	**Application**	**Example**
re	`z.A.re = 1`	Refer to (7.2.2)
im	`z.A.im = 2`	Refer to (7.2.2)
type	`z.A.type = 'point'`	
argument	`z.A.argument $\approx$ 0.78539816339745`	Refer to (8.1.1)
modulus	`z.A.modulus` $\approx$ `2.2360...` $=\sqrt{5}$	Refer to (8.1.1)

8.1.1 Example:point attributes

```
\begin{tkzelements}
   z.M = point: new (1,2)
\end{tkzelements}

\begin{tikzpicture}[scale = 1]
\pgfkeys{/pgf/number format/.cd,std,precision=2}
\let\pmpn\pgfmathprintnumber
\tkzDefPoints{2/4/M,2/0/A,0/0/O,0/4/B}
\tkzLabelPoints(O)
\tkzMarkAngle[fill=gray!30,size=1](A,O,M)
\tkzLabelAngle[pos=1,right](A,O,M){%
$\theta \approx \pmpn{\tkzUseLua{z.M.argument}}$ rad}
\tkzDrawSegments(O,M)
\tkzLabelSegment[above,sloped](O,M){%
$|z_M| =\sqrt{5}\approx \pmpn{\tkzUseLua{z.M.modulus}}$ cm}
\tkzLabelPoint[right](M){$M : z_M = 1 + 2i$}
\tkzDrawPoints(M,A,O,B)
\tkzPointShowCoord(M)
\tkzLabelPoint[below,teal](A){$\tkzUseLua{z.M.re}$}
\tkzLabelPoint[left,teal](B){$\tkzUseLua{z.M.im}$}
\tkzDrawSegments[->,add = 0 and 0.25](O,B O,A)
\end{tikzpicture}
```

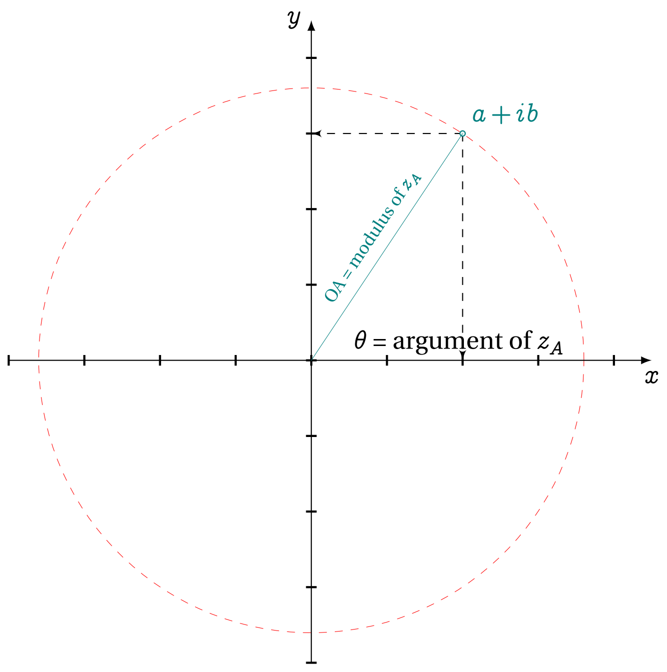
Attributes of `z.M`

- `z.M.re` = 1
- `z.M.im` = 2
- `z.M.type` = 'point'
- `z.M.argument` = $\theta \approx 1.11$ rad
- `z.M.modulus` = $|z_M| = \sqrt{5} \approx 2.24$ cm



8.1.2 Argand diagram

```
\begin{tkzelements}
   z.A = point : new ( 2 , 3 )
   z.O = point : new ( 0 , 0 )
   z.I = point : new ( 1 , 0 )
\end{tkzelements}
\hspace{\fill}\begin{tikzpicture}
   \tkzGetNodes
   \tkzInit[xmin=-4,ymin=-4,xmax=4,ymax=4]
   \tkzDrawCircle[dashed,red](O,A)
   \tkzPointShowCoord(A)
   \tkzDrawPoint(A)
   \tkzLabelPoint[above right](A){\normalsize $a+ib$}
   \tkzDrawX\tkzDrawY
   \tkzDrawSegment(O,A)
   \tkzLabelSegment[above,anchor=south,sloped](O,A){ OA = modulus of $z_A$}
  \tkzLabelAngle[anchor=west,pos=.5](I,O,A){$\theta$ = argument of $z_A$}
\end{tikzpicture}
```



8.2 Methods of the class point

The methods described in the following table are standard and can be found in most of the examples at the end of this documentation. The result of the different methods presented in the following table is a **`point`**. Refer to section (22.3) for the metamethods.

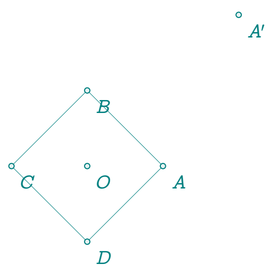
Table 2: Functions & Methods of the class point.

Functions	Application	
`new(r,r)`	`z.A = point : new(1,2)`	Refer to (8.2.4)
`polar (d,an)`	`z.A = point : polar(1,math.pi/3)`	Refer to (23.7)
`polar_deg (d,an)`	an in deg	polar coordinates an deg
Methods	**Application**	
Points		
`north(r)`	`r` distance to the point (1 if empty)	Refer to (23.43) ; 7.2.2)
`south(r)`		
`east(r)`		
`west(r)`		
`normalize()`	`z.b = z.a: normalize ()`	Refer to (8.2.4)
`get_points (obj)`	retrieves points from the object	
`orthogonal (d)`	`z.B=z.A:orthogonal(d)`	$\overrightarrow{OB} \perp \overrightarrow{OA}$ and $OB = d$
`at ()`	`z.X = z.B : at (z.A)`	$\overrightarrow{OB} = \overrightarrow{AX}$ and $OB = d$
Transformations		
`symmetry(obj)`	obj : point, line, etc.	Refer to (8.2.9)
`rotation(an , obj)`	point, line, etc.	Refer to (8.2.8)
`homothety(r,obj)`	`z.c = z.a : homothety (2,z.b)`	Refer to (23.48)
Misc.		
`print()`	displays the affix of the point	Refer to (8.2.9)

8.2.1 Example: method north (d)

This function defines a point located on a vertical line passing through the given point. This function is useful if you want to report a certain distance (Refer to the following example). If `d` is absent then it is considered equal to 1.

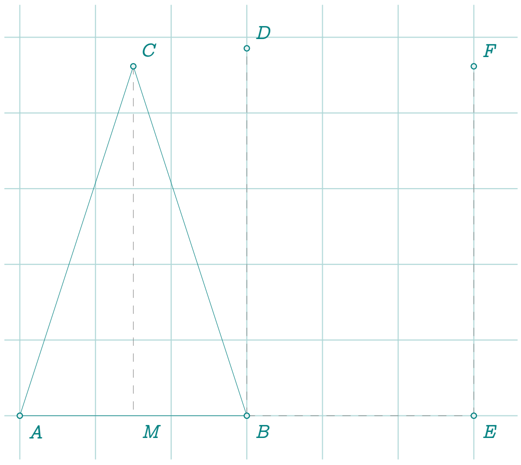
```
\begin{tkzelements}
   z.O   = point : new ( 0, 0 )
   z.A   = z.O : east ()
   z.Ap  = z.O : east (2) : north (2)
   z.B   = z.O : north ()
   z.C   = z.O : west ()
   z.D   = z.O : south ()
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C,D)
   \tkzDrawPoints(A,B,C,D,O,A')
\end{tikzpicture}
```



8.2.2 Length transfer

Use of `north` and `east` functions linked to points, to transfer lengths, Refer to (20.1)

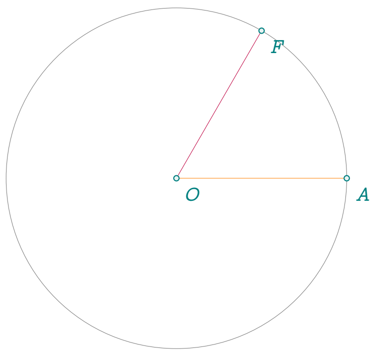
```
\begin{tkzelements}
   z.A = point : new ( 0 , 0 )
   z.B = point : new ( 3 , 0 )
   L.AB = line : new ( z.A , z.B )
   T.ABC =   L.AB : sublime ()
   z.C = T.ABC.pc
   z.D = z.B: north (length(z.B,z.C))
   z.E = z.B: east (L.AB.length)
   z.M = L.AB.mid
   z.F = z.E : north (length(z.C,z.M))
\end{tkzelements}
\begin{tikzpicture}[gridded]
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C)
   \tkzDrawSegments[gray,dashed](B,D B,E E,F C,M)
   \tkzDrawPoints(A,...,F)
   \tkzLabelPoints(A,B,E,M)
   \tkzLabelPoints[above right](C,D,F)
\end{tikzpicture}
```



8.2.3 Example: method polar

This involves defining a point using its modulus and argument.

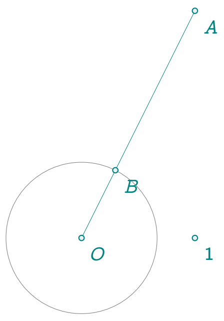
```
\begin{tkzelements}
   z.O     = point:   new  (0, 0)
   z.A     = point:   new  (3, 0)
   z.F     = point:   polar (3, math.pi/3)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(O,A)
   \tkzDrawSegments[new](O,A)
   \tkzDrawSegments[purple](O,F)
   \tkzDrawPoints(A,O,F)
   \tkzLabelPoints[below right=6pt](A,O,F)
\end{tikzpicture}
```



8.2.4 Method normalize ()

The result is a point located between the origin and the initial point at a distance of 1 from the origin.

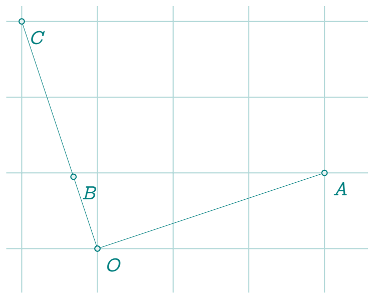
```
\begin{tkzelements}
   scale = 1.5
   z.O = point : new (0,0)
   z.A = point : new (1,2)
   z.B = z.A : normalize ()
   z.I = point : new (1,0)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawSegment(O,A)
   \tkzDrawCircle(O,B)
   \tkzDrawPoints(O,A,B,I)
   \tkzLabelPoints(O,A,B)
   \tkzLabelPoint[below right](I){$1$}
\end{tikzpicture}
```



8.2.5 Orthogonal (d) method

Let O be the origin of the plane. The "orthogonal (d)" method is used to obtain a point B from a point A such that $\overrightarrow{OB} \perp \overrightarrow{OA}$ with $OB = OA$ if d is empty, otherwise $OB = d$.

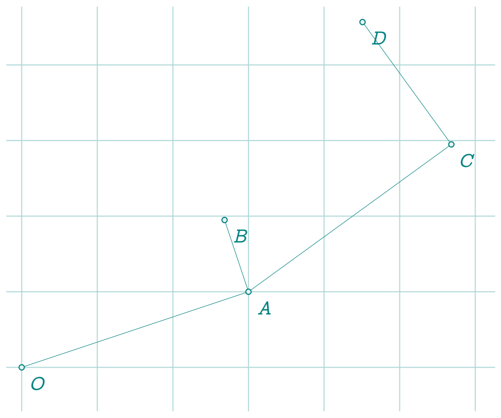
```
\begin{tkzelements}
  z.A = point : new (  3 , 1  )
  z.B = z.A : orthogonal (1)
  z.O = point : new ( 0,0  )
  z.C = z.A : orthogonal ()
\end{tkzelements}
\begin{tikzpicture}[gridded]
  \tkzGetNodes
  \tkzDrawSegments(O,A O,C)
  \tkzDrawPoints(O,A,B,C)
  \tkzLabelPoints[below right](O,A,B,C)
\end{tikzpicture}
```



8.2.6 at method

This method is complementary to the previous one, so you may not wish to have $\overrightarrow{OB} \perp \overrightarrow{OA}$ but $\overrightarrow{AB} \perp \overrightarrow{OA}$.

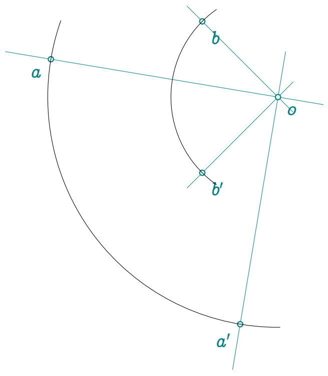
```
\begin{tkzelements}
  z.A = point : new (  3 , 1  )
  z.B = z.A : orthogonal (1)
  z.O = point : new ( 0,0  )
  -- z.B = z.B : at (z.A) -- or
  z.B = z.A : orthogonal (1) : at (z.A)
  z.C = z.A+z.B
  z.D =(z.C-z.A):orthogonal(2) : at (z.C)
\end{tkzelements}
\begin{tikzpicture}[gridded]
  \tkzGetNodes
  \tkzLabelPoints[below right](O,A,B,C,D)
  \tkzDrawSegments(O,A A,B A,C C,D)
  \tkzDrawPoints(O,A,B,C,D)
\end{tikzpicture}
```



8.2.7 Example: rotation of points

The arguments are the angle of rotation in radians, and here a list of points.

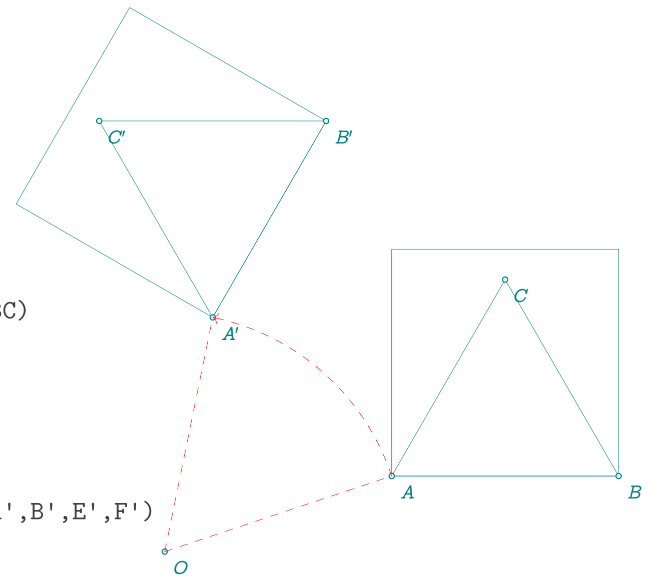
```
\begin{tkzelements}
  z.a       = point:  new(0, -1)
  z.b       = point:  new(4, 0)
  z.o       = point:  new(6, -2)
  z.ap,z.bp = z.o : rotation (math.pi/2,z.a,z.b)
\end{tkzelements}
      \begin{tikzpicture}
      \tkzGetNodes
      \tkzDrawLines(o,a o,a' o,b o,b')
      \tkzDrawPoints(a,a',b,b',o)
      \tkzLabelPoints(b,b',o)
      \tkzLabelPoints[below left](a,a')
      \tkzDrawArc(o,a)(a')
      \tkzDrawArc(o,b)(b')
      \end{tikzpicture}
```



8.2.8 Object rotation

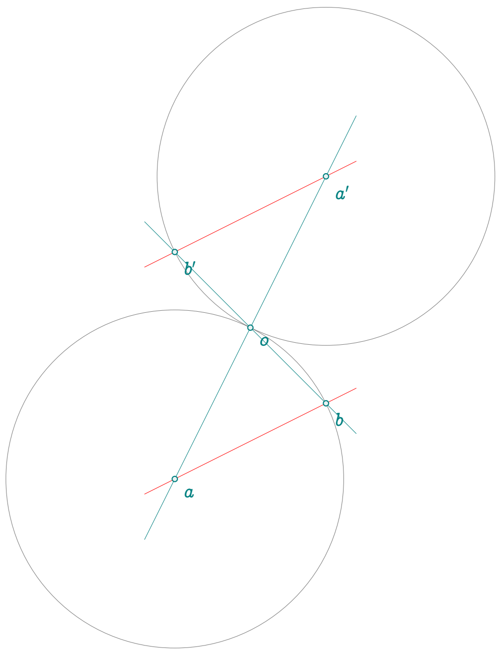
Rotate a triangle by an angle of $\pi/6$ around O.

```
\begin{tkzelements}
   z.O   = point : new ( -1 , -1 )
   z.A   = point : new ( 2 , 0 )
   z.B   = point : new ( 5 , 0 )
   L.AB  = line : new (z.A,z.B)
   T.ABC = L.AB : equilateral ()
   S.fig = L.AB : square ()
   _,_,z.E,z.F = get_points (   S.fig   )
   S.new = z.O : rotation (math.pi/3,S.fig)
   _,_,z.Ep,z.Fp = get_points (   S.new   )
   z.C = T.ABC.pc
   T.ApBpCp = z.O : rotation (math.pi/3,T.ABC)
   z.Ap,z.Bp,z.Cp = get_points ( T.ApBpCp)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C A',B',C' A,B,E,F A',B',E',F')
   \tkzDrawPoints (A,B,C,A',B',C',O)
    \tkzLabelPoints (A,B,C,A',B',C',O)
    \tkzDrawArc[delta=0,->](O,A)(A')
\end{tikzpicture}
```



8.2.9 Object symmetry

```
\begin{tkzelements}
    z.a = point:  new(0,-1)
    z.b = point:  new(2, 0)
    L.ab = line : new (z.a,z.b)
    C.ab = circle : new (z.a,z.b)
    z.o = point:  new(1,1)
    z.ap,z.bp =  get_points (z.o: symmetry (C.ab))
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawCircles(a,b a',b')
\tkzDrawLines(a,a' b,b')
\tkzDrawLines[red](a,b a',b')
\tkzDrawPoints(a,a',b,b',o)
\tkzLabelPoints(a,a',b,b',o)
\end{tikzpicture}
```



9 Class line

9.1 Attributes of a line

Writing `L.AB = line: new (z.A,z.B)` creates an object of the class **`line`** (the notation is arbitrary for the moment). Geometrically, it represents both the line passing through the points A and B as the segment $[AB]$. Thus, we can use the midpoint of `L.AB`, which is, of course, the midpoint of the segment $[AB]$. This medium is obtained with `L.AB.mid`. Note that `L.AB.pa = z.A` and `L.AB.pb = z.B`. Finally, if a line L is the result of a method, you can obtain the points with `z.A,z.B = get_points (L)` or with the previous remark.

Creation `L.AB = line : new ( z.A , z.B )`

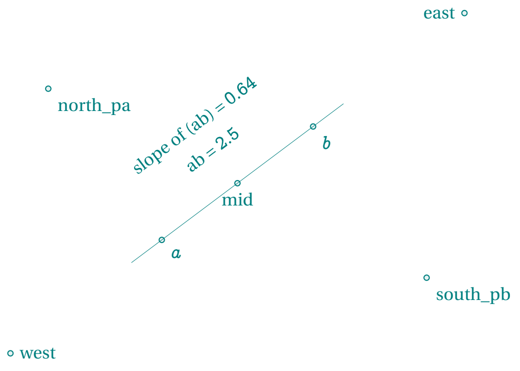
The attributes are :

Table 3: Line attributes.

Attributes	Application	
pa	First point of the segment	`z.A = L.AB.pa`
pb	Second point of the segment	
type	Type is 'line'	`L.AB.type = 'line'`
mid	Middle of the segment	`z.M = L.AB.mid`
slope	Slope of the line	Refer to (9.1.1)
length	`l = L.AB.length`	Refer to (20.6 ; 9.1.1)
north_pa		Refer to (9.1.1)
north_pb		
south_pa		
south_pb		Refer to (9.1.1)
east		
west		
vec	`V.AB = L.AB.vec`	defines $\overrightarrow{AB}$ Refer to (18)

9.1.1 Example: attributes of class line

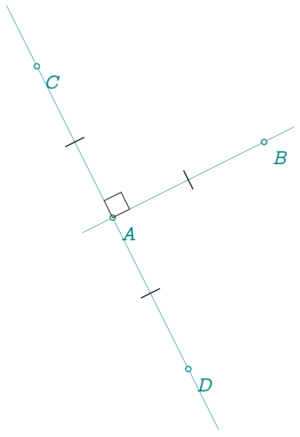
```
\begin{tkzelements}
  scale  = .5
   z.a   = point: new (1, 1)
   z.b   = point: new (5, 4)
   L.ab  = line : new (z.a,z.b)
   z.m   = L.ab.mid
   z.w   = L.ab.west
   z.e   = L.ab.east
   z.r   = L.ab.north_pa
   z.s   = L.ab.south_pb
   sl    = L.ab.slope
   len   = L.ab.length
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPoints(a,b,m,e,r,s,w)
   \tkzLabelPoints(a,b,e,r,s,w)
   \tkzLabelPoints[above](m)
   \tkzDrawLine(a,b)
   \tkzLabelSegment[sloped](a,b){ab = \tkzUseLua{len}}
   \tkzLabelSegment[above=12pt,sloped](a,b){slope of (ab) = \tkzUseLua{sl}}
\end{tikzpicture}
```



9.1.2 Method new and line attributes

The notation can be `L` or `L.AB` or `L.euler`. The notation is actually free. `L.AB` can also represent the segment. With `L.AB = line : new (z.A,z.B)`, a line is defined.

```
\begin{tkzelements}
   z.A   = point : new (1,1)
   z.B   = point : new (3,2)
   L.AB  = line : new (z.A,z.B)
   z.C   = L.AB.north_pa
   z.D   = L.AB.south_pa
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines(A,B C,D)
   \tkzDrawPoints(A,...,D)
   \tkzLabelPoints(A,...,D)
   \tkzMarkRightAngle(B,A,C)
   \tkzMarkSegments(A,C A,B A,D)
\end{tikzpicture}
```



9.2 Methods of the class line

Here's the list of methods for the **line** object. The results can be real numbers, points, lines, circles or triangles. The triangles obtained are similar to the triangles defined below.

Table 4: Methods of the class line.(part 1)

Methods	Comments	
`new(pt, pt)`	`L.AB = line : new(z.A,z.B)`	Create line (AB) ; Refer to (10.2.1)
Points		
`gold_ratio ()`	`z.C=L.AB : gold_ratio()`	Refer to (23.21 ; 3.3 ; 23.8)
`normalize ()`	`z.C=L.AB : normalize()`	AC =1 and $C \in (AB)$ Refer to (9.2.7)
`normalize_inv ()`	`z.C=L.AB : normalize_inv()`	CB=1 and $C \in (AB)$
`barycenter (r,r)`	`z.C=L.AB : barycenter (1,2)`	Refer to (9.2.8)
`point (r)`	`z.C=L.AB : point (2)`	$\overrightarrow{AC} = 2\overrightarrow{AB}$ Refer to (23.14 ; 9.2.6)
`midpoint ()`	`z.M=L.AB : midpoint ()`	better is `z.M = L.AB.mid`
`harmonic_int (pt)`	`z.D=L.AB : harmonic_int (z.C)`	Refer to (23.8)
`harmonic_ext (pt)`	`z.D=L.AB : harmonic_ext (z.C)`	Refer to (23.8)
`harmonic_both (r)`	`z.C,z.D=L.AB : harmonic_both(`φ`)`	20.2
`_east(d)`	`z.M=L.AB : _east(2)`	`BM = 2` A, B, M aligned
`_west(d)`	`z.M=L.AB : _east(2)`	`BM = 2` A, B, M aligned
`_north_pa(d)`	`z.M=L.AB: _north_pa(2)`	AM=2 $AM \perp AB$; $\overrightarrow{AB}, \overrightarrow{AM}$ counterclockwise
`_south_pa(d)`	`z.M=L.AB:_south_pa(2)`	AM=2; $AM \perp AB$; $\overrightarrow{AB}, \overrightarrow{AM}$ clockwise
`_north_pb(d)`	`z.M=L.AB:_north_pb(2)`	BM=2; $BM \perp BA$; $\overrightarrow{BA}, \overrightarrow{BM}$ clockwise
`_south_pb(d)`	`z.M=L.AB:_south_pb(2)`	BM=2; $BM \perp BA$; $\overrightarrow{AB}, \overrightarrow{AM}$ counterclockwise
`report(d,pt)`	`z.M=L.AB:report(2,z.N)`	MN=2; $AB \parallel MN$; Refer to ex. (9.2.1)
Lines		
`ll_from ( pt )`	`L.CD=L.AB: ll_from(z.C)`	$(CD) \parallel (AB)$
`ortho_from ( pt )`	`L.CD=L.AB: ortho_from(z.C)`	$(CD) \perp (AB)$
`mediator ()`	`L.uv=L.AB: mediator()`	(u,v)mediator of (A,B)
Triangles		
`equilateral (<swap>)`	`T.ABC=L.AB:equilateral()`	$(\overrightarrow{AB}, \overrightarrow{AC}) > 0$ or < 0 with swap [a]
`isosceles (an<,swap>)`	`T.ABC=L.AB:isosceles(math.pi/6)`	
`two_angles (an,an)`	`T.ABC=L.AB:two_angles(an,an)`	note [b] Refer to (9.2.2)
`school ()`	30°,60°, 90°	
`sss (r,r)`	$AC = r\ BC = r$	
`as (r,an)`	$AC = r\ \widehat{BAC} = an$	
`sa (r,an)`	$AC = r\ \widehat{ABC} = an$	
Sacred triangles		
`gold (<swap>)`	`T.ABC=L.AB:gold()`	right in B and $AC = \varphi \times AB$
`euclide (<swap>)`	`T.ABC=L.AB:euclide()`	$AB = AC$; $(\overrightarrow{AB}, \overrightarrow{AC}) = \pi/5$
`golden (<swap>)`	`T.ABC=L.AB:golden()`	$(\overrightarrow{AB}, \overrightarrow{AC}) = 2 \times \pi/5$
`divine ()`		
`egyptian ()`		
`cheops ()`		
Squares		
`square ()`	`S.AB=L.AB : square ()`	create a square `S.AB`.[c]

a Triangles are defined in the direct sense of rotation, unless the "swap" option is present.
b The given side is between the two angles
c `_,_,z.C,z.D = get_points(S.AB)`

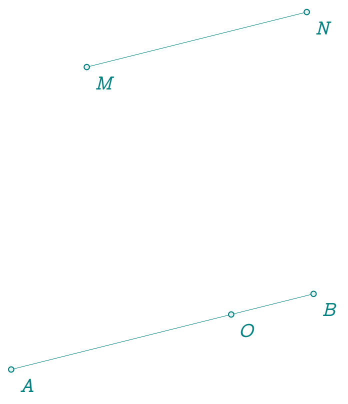
Table 5: Methods of the class line.(part 2)

Methods	Comments	
Circles		
`circle ()`	`C.AB = L.AB : circle ()`	center pa through pb
`circle_swap ()`	`C.BA = L.AB : circle_swap ()`	center pb through pa
`apollonius (r)`	`C.apo = L.AB : apollonius (2)`	Ensemble des points tq. `MA/MB = 2`
Transformations		
`reflection ( obj )`	`new obj = L.AB : reflection (obj`	
`translation ( obj )`	`new obj = L.AB : translation (obj)`	
`projection ( obj )`	`z.H = L.AB : projection (z.C)`	$CH \perp (AB)$ and $H \in (AB)$
Miscellaneous		
`distance (pt)`	`d = L.Ab : distance (z.C)`	Refer to 9.2.13
`in_out (pt)`	`b = L.AB: in_out(z.C)`	b=true if $C \in (AB)$
`slope ()`	`a = L.AB : slope()`	better is L.AB.slope
`in_out_segment (pt)`	`b = L.AB : in_out_segment(z.C)`	b=true if $C \in [AB]$

9.2.1 Method report

`report (d,pt)` If the point is absent, the transfer is made from the first point that defines the line.

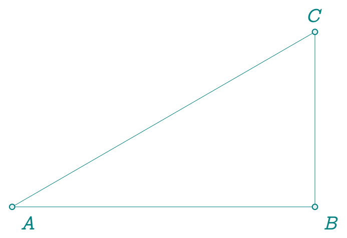
```
\begin{tkzelements}
z.A  = point : new (1,-1)
z.B  = point : new (5,0)
L.AB = line : new ( z.A , z.B )
z.M  = point : new (2,3)
z.N  = L.AB :  report (3,z.M)
z.O  = L.AB :  report (3)
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawSegments(A,B M,N)
\tkzDrawPoints(A,B,M,N,O)
\tkzLabelPoints(A,B,M,N,O)
\end{tikzpicture}
```



9.2.2 Triangle with two_angles

The angles are on either side of the given segment

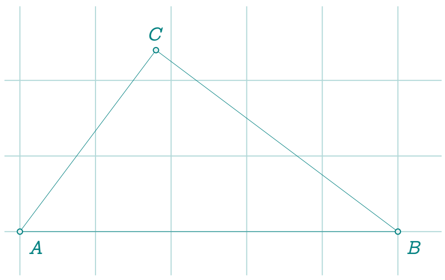
```
\begin{tkzelements}
   z.A   = point : new ( 0 , 0 )
   z.B   = point : new ( 4 , 0 )
   L.AB  = line : new ( z.A , z.B )
   T.ABC = L.AB : two_angles (math.pi/6,math.pi/2)
   z.C   = T.ABC.pc
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C)
   \tkzDrawPoints(A,B,C)
   \tkzLabelPoints(A,B)
    \tkzLabelPoints[above](C)
\end{tikzpicture}
```



9.2.3 Triangle with three given sides

In the following example, a small difficulty arises. The given lengths are not affected by scaling, so it's necessary to use the value (r) function, which will modify the lengths according to the scale.

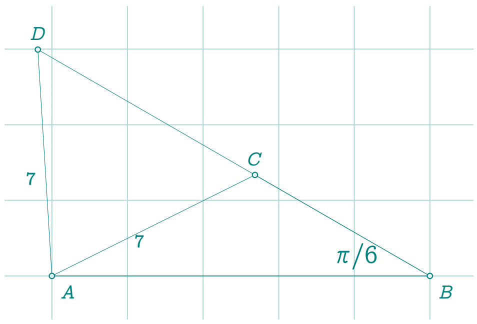
```
\begin{tkzelements}
   z.A = point : new ( 0 , 0 )
   z.B = point : new ( 5 , 0 )
   L.AB = line : new ( z.A , z.B )
   T.ABC =   L.AB : sss (value(3),value(4))
   z.C = T.ABC.pc
\end{tkzelements}
\begin{tikzpicture}[gridded]
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C)
   \tkzDrawPoints(A,B,C)
   \tkzLabelPoints(A,B)
   \tkzLabelPoints[above](C)
\end{tikzpicture}
```



9.2.4 Triangle with side between side and angle

In some cases, two solutions are possible.

```
\begin{tkzelements}
   scale =1
   z.A = point : new ( 0 , 0 )
   z.B = point : new ( 5 , 0 )
   L.AB = line : new ( z.A , z.B )
   T.ABC,T.ABD =   L.AB : ssa (value(3),math.pi/6)
   z.C = T.ABC.pc
   z.D = T.ABD.pc
\end{tkzelements}
\begin{tikzpicture}[gridded]
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C A,B,D)
   \tkzDrawPoints(A,B,C,D)
   \tkzLabelPoints(A,B)
   \tkzLabelPoints[above](C,D)
   \tkzLabelAngle[teal](C,B,A){$\pi/6$}
   \tkzLabelSegment[below left](A,C){$7$}
   \tkzLabelSegment[below left](A,D){$7$}
\end{tikzpicture}
```



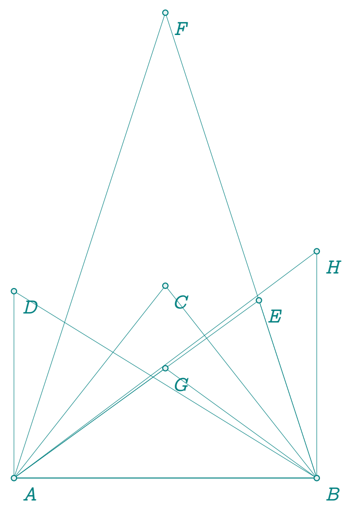
9.2.5 About sacred triangles

The side lengths are proportional to the lengths given in the table. They depend on the length of the initial segment.

Table 6: Sacred triangles.

Name	definition
`gold (<swap>)`	Right triangle with $a = \varphi$, $b = 1$ and $c = \sqrt{\varphi}$
`golden (<swap>)`	Right triangle $b = \varphi$ $c = 1$; half of gold rectangle
`divine ()`	Isosceles $a = \varphi$, $b = c = 1$ and $\beta = \gamma = \pi/5$
`pythagoras ()`	$a = 5$, $b = 4$, $c = 3$ and other names: isis or egyptian
`sublime ()`	Isosceles $a = 1$, $b = c = \varphi$ and $\beta = \gamma = 2\pi/5$; other name: euclid
`cheops ()`	Isosceles $a = 2$, $b = c = \varphi$ and height = $\sqrt{\varphi}$

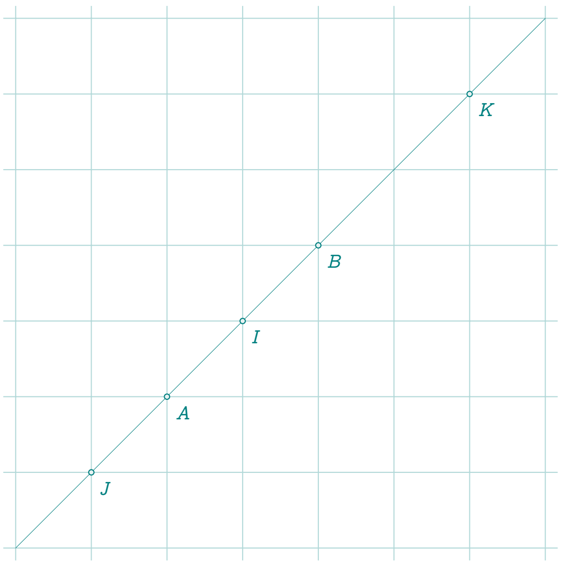
```
\begin{tkzelements}
   z.A = point : new ( 0 , 0 )
   z.B = point : new ( 4 , 0 )
   L.AB = line : new ( z.A , z.B )
   T.ABC = L.AB : cheops ()
   z.C = T.ABC.pc
   T.ABD = L.AB : gold ()
   z.D = T.ABD.pc
   T.ABE = L.AB : euclide ()
   z.E = T.ABE.pc
   T.ABF = L.AB : golden ()
   z.F = T.ABF.pc
   T.ABG = L.AB : divine ()
   z.G = T.ABG.pc
   T.ABH = L.AB : pythagoras ()
   z.H = T.ABH.pc
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C A,B,D A,B,E A,B,F A,B,G A,B,H)
   \tkzDrawPoints(A,...,H)
   \tkzLabelPoints(A,...,H)
\end{tikzpicture}
```



9.2.6 Method `point`

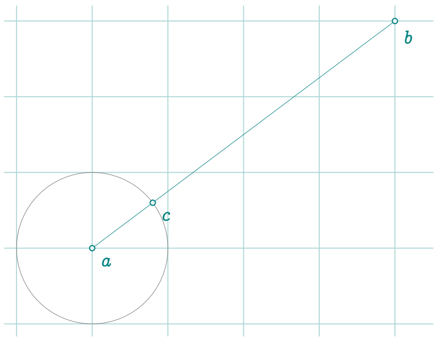
This method is very useful. It allows you to place a point on the line under consideration. If `r = 0` then the point is pa, if `r = 1` it's pb.
If `r = .5` the point obtained is the midpoint of the segment. `r` can be negative or greater than 1.
This method exists for all objects except quadrilaterals.

```
\begin{tkzelements}
   z.A = point : new (-1,-1)
   z.B = point : new (1,1)
   L.AB = line : new (z.A,z.B)
   z.I = L.AB : point (0.5)
   z.J = L.AB : point (-0.5)
   z.K = L.AB : point (2)
\end{tkzelements}
\begin{tikzpicture}[gridded]
\tkzGetNodes
   \tkzDrawLine(J,K)
   \tkzDrawPoints(A,B,I,J,K)
   \tkzLabelPoints(A,B,I,J,K)
 \end{tikzpicture}
```



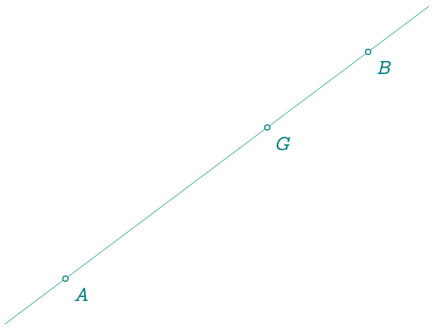
9.2.7 Normalize

```
\begin{tkzelements}
   z.a = point: new (1, 1)
   z.b = point: new (5, 4)
   L.ab = line : new (z.a,z.b)
   z.c   = L.ab : normalize ()
\end{tkzelements}

\begin{tikzpicture}[gridded]
\tkzGetNodes
\tkzDrawSegments(a,b)
\tkzDrawCircle(a,c)
\tkzDrawPoints(a,b,c)
\tkzLabelPoints(a,b,c)
\end{tikzpicture}
```



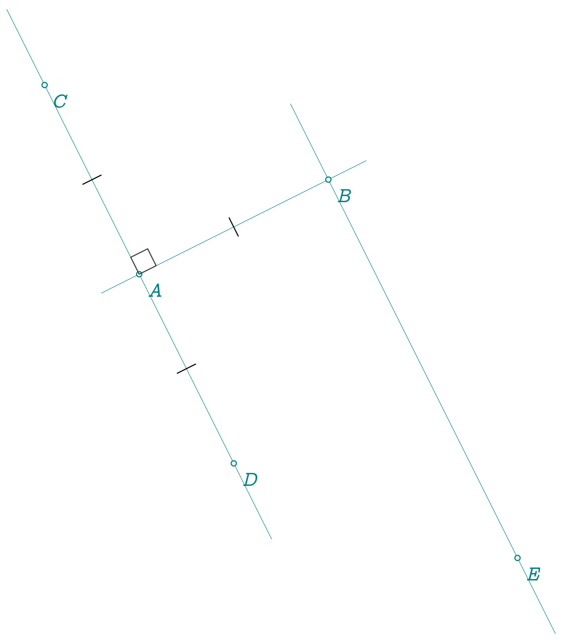
9.2.8 Barycenter with a line

```
\begin{tkzelements}
   z.A = point : new ( 0 , -1 )
   z.B = point : new ( 4 , 2 )
   L.AB = line : new ( z.A , z.B )
   z.G = L.AB : barycenter (1,2)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLine(A,B)
   \tkzDrawPoints(A,B,G)
   \tkzLabelPoints(A,B,G)
\end{tikzpicture}
```



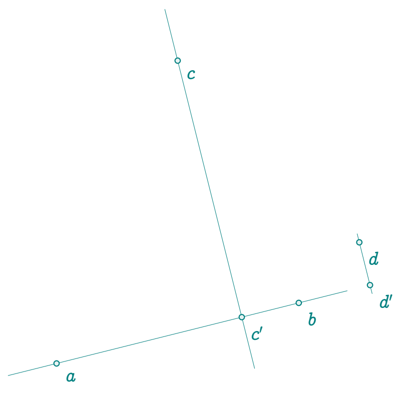
9.2.9 Example: new line from a defined line

```
\begin{tkzelements}
   scale = 1.25
   z.A   = point : new (1,1)
   z.B   = point : new (3,2)
   L.AB  = line : new (z.A,z.B)
   z.C   = L.AB.north_pa
   z.D   = L.AB.south_pa
   L.CD  = line : new (z.C,z.D)
   _,z.E   = get_points ( L.CD: ll_from (z.B))
   -- z.E   = L2.pb
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines(A,B C,D B,E)
   \tkzDrawPoints(A,...,E)
   \tkzLabelPoints(A,...,E)
   \tkzMarkRightAngle(B,A,C)
   \tkzMarkSegments(A,C A,B A,D)
\end{tikzpicture}
```



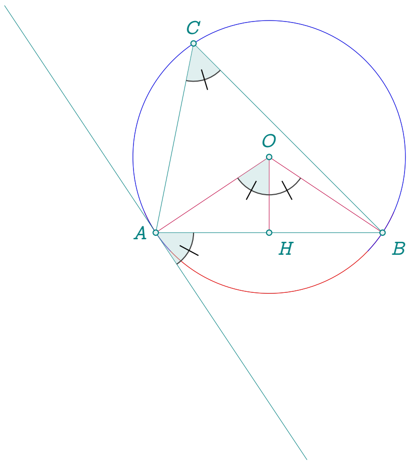
9.2.10 Example: projection of several points

```
\begin{tkzelements}
   scale     = .8
   z.a       = point:  new (0, 0)
   z.b       = point:  new (4, 1)
   z.c       = point:  new (2, 5)
   z.d       = point:  new (5, 2)
   L.ab      = line:   new (z.a,z.b)
   z.cp,z.dp = L.ab:  projection(z.c,z.d)
\end{tkzelements}
 \begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines(a,b c,c' d,d')
   \tkzDrawPoints(a,...,d,c',d')
   \tkzLabelPoints(a,...,d,c',d')
 \end{tikzpicture}
```



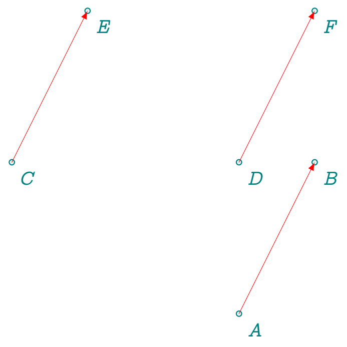
9.2.11 Example: combination of methods

```
\begin{tkzelements}
   z.A      = point: new (0 , 0)
   z.B      = point: new (6 , 0)
   z.C      = point: new (1 , 5)
   T.ABC    = triangle: new (z.A,z.B,z.C)
   L.AB     = T.ABC.ab
   z.O      = T.ABC.circumcenter
   C.OA     = circle: new (z.O,z.A)
   z.H      = L.AB: projection (z.O)
   L.ab     = C.OA: tangent_at (z.A)
   z.a,z.b = L.ab.pa,L.ab.pb
  -- or z.a,z.b  = get_points (L.ab)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawCircle(O,A)
   \tkzDrawSegments[purple](O,A O,B O,H)
   \tkzDrawArc[red](O,A)(B)
   \tkzDrawArc[blue](O,B)(A)
   \tkzDrawLine[add = 2 and 1](A,a)
   \tkzFillAngles[teal!30,opacity=.4](A,C,B b,A,B A,O,H)
   \tkzMarkAngles[mark=|](A,C,B b,A,B A,O,H H,O,B)
   \tkzDrawPoints(A,B,C,H,O)
   \tkzLabelPoints(B,H)
   \tkzLabelPoints[above](O,C)
   \tkzLabelPoints[left](A)
\end{tikzpicture}
```



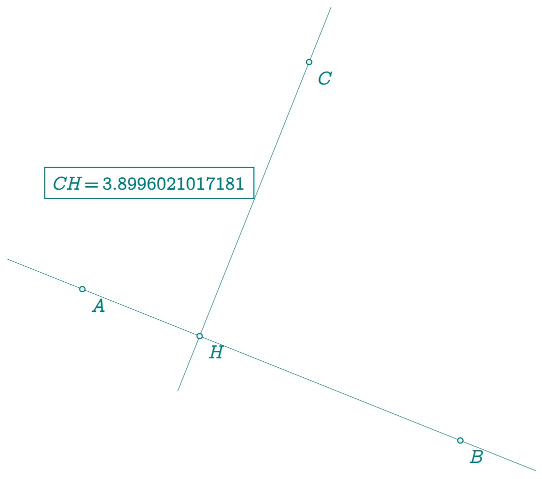
9.2.12 Example: translation

```
\begin{tkzelements}
   z.A  = point:  new (0,0)
   z.B  = point:  new (1,2)
   z.C  = point:  new (-3,2)
   z.D  = point:  new (0,2)
   L.AB = line : new (z.A,z.B)
   z.E,z.F = L.AB : translation (z.C,z.D)
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPoints(A,...,F)
\tkzLabelPoints(A,...,F)
\tkzDrawSegments[->,red,> =latex](C,E D,F A,B)
\end{tikzpicture}
```



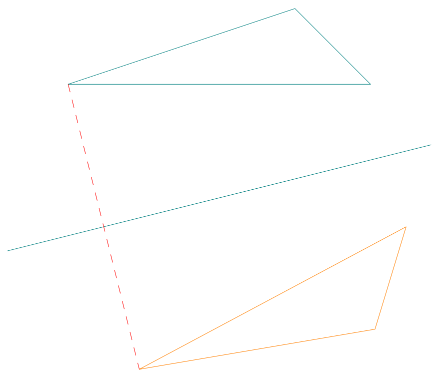
9.2.13 Example: distance and projection

```
\begin{tkzelements}
   z.A    = point : new (0 , 0)
   z.B    = point : new (5 , -2)
   z.C    = point : new (3 , 3)
   L.AB   = line : new (z.A,z.B)
   d      = L.AB : distance (z.C)
   z.H    = L.AB : projection (z.C)
\end{tkzelements}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawLines(A,B C,H)
  \tkzDrawPoints(A,B,C,H)
  \tkzLabelPoints(A,B,C,H)
  \tkzLabelSegment[above left,
  draw](C,H){$CH = \tkzUseLua{d}$}
\end{tikzpicture}
```



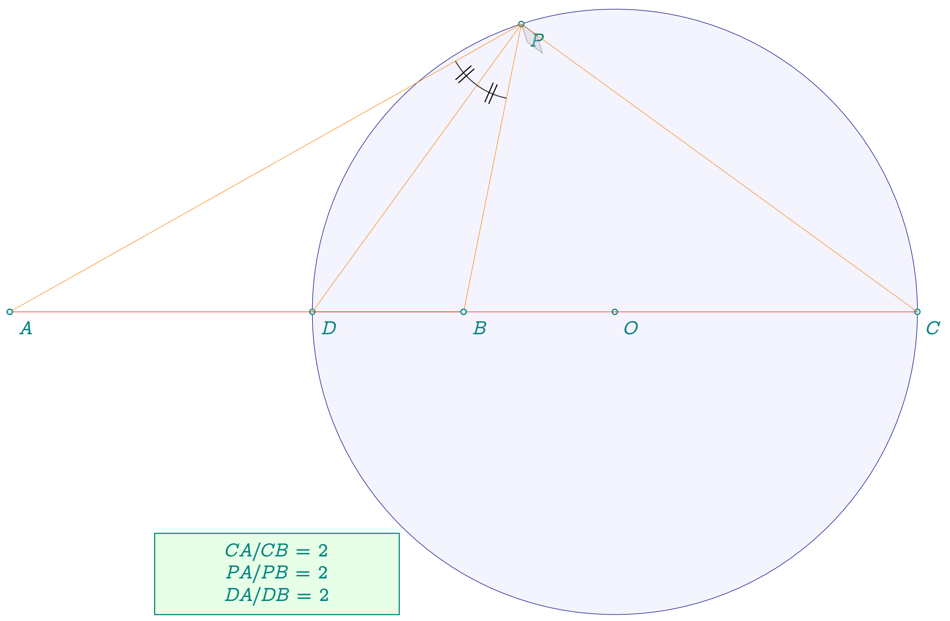
9.2.14 Reflection of object

```
\begin{tkzelements}
   z.A = point : new ( 0 , 0 )
   z.B = point : new ( 4 , 1 )
   z.E = point : new ( 0 , 2 )
   z.F = point : new ( 3 , 3 )
   z.G = point : new ( 4 , 2 )
   L.AB = line : new ( z.A , z.B )
   T.EFG = triangle : new (z.E,z.F,z.G)
   T.new = L.AB : reflection (T.EFG)
   z.Ep,z.Fp,z.Gp = get_points(T.new)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLine(A,B)
   \tkzDrawPolygon(E,F,G)
   \tkzDrawPolygon[new](E',F',G')
   \tkzDrawSegment[red,dashed](E,E')
\end{tikzpicture}
```



9.3 Apollonius circle MA/MB = k

```
\begin{tkzelements}
   z.A = point : new ( 0 , 0 )
   z.B = point : new ( 6 , 0 )
   L.AB =line: new (z.A,z.B)
   C.apo = L.AB : apollonius (2)
   z.O,z.C = get_points (   C.apo    )
   z.D = C.apo : antipode (z.C)
   z.P = C.apo : point  (0.30)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzFillCircle[blue!20,opacity=.2](O,C)
   \tkzDrawCircle[blue!50!black](O,C)
   \tkzDrawPoints(A,B,O,C,D,P)
   \tkzLabelPoints[below right](A,B,O,C,D,P)
   \tkzDrawSegments[orange](P,A P,B P,D B,D P,C)
   \tkzDrawSegments[red](A,C)
   \tkzDrawPoints(A,B)
   \tkzLabelCircle[draw,fill=green!10,%
      text width=3cm,text centered,left=24pt](O,D)(60)%
     {$CA/CB=2$\\$PA/PB=2$\\$DA/DB=2$}
   \tkzMarkRightAngle[opacity=.3,fill=lightgray](O,P,C)
   \tkzMarkAngles[mark=||](A,P,D D,P,B)
\end{tikzpicture}
```



Remark: \tkzUseLua{length(z.P,z.A)/length(z.P,z.B)} = 2.0

10 Class circle

10.1 Attributes of a circle

This class is defined by two points: the center and a point through which the circle passes

Creation `C.OA = circle: new (z.O,z.A)`

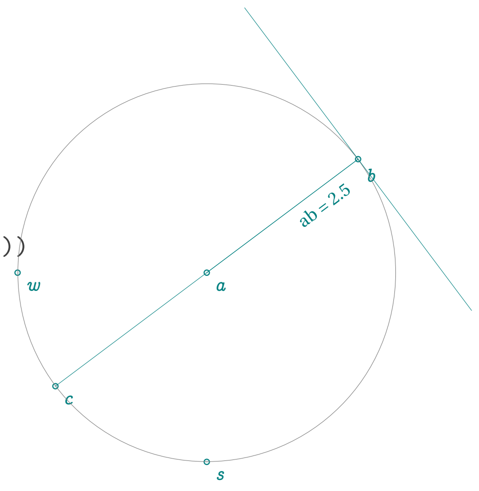
Table 7: Circle attributes.

Attributes	Application	
center	`z.A = C.AB.center`	
through	`z.B = C.AB.through`	
type	`C.AB.type`	`C.OA.type = 'circle'`
radius	`C.AB.radius`	`r = C.OA.radius` r real number
north	`C.AB.north`	`z.N = C.OA.north`
south	`C.AB.south`	`z.S = C.OA.south`
east	`C.AB.east`	`z.E = C.OA.east`
west	`C.AB.west`	`z.W = C.OA.west`
opp	`z.Ap = C.AB.opp`	Refer to (10.1.1)
ct	`L = C.AB.ct`	Refer to (10.1.1)

10.1.1 Example: circle attributes

Three attributes are used (south, west, radius).

```
\begin{tkzelements}
   scale = .5
   z.a   = point: new (1, 1)
   z.b   = point: new (5, 4)
   C.ab  = circle : new (z.a,z.b)
   z.s   = C.ab.south
   z.w   = C.ab.west
   r     = C.ab.radius
   z.c   = C.ab.opp
   z.r,z.t = get_points (C.ab.ct : ortho_from (z.b))
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPoints(a,b,c,s,w)
\tkzLabelPoints(a,b,c,s,w)
\tkzDrawCircle(a,b)
\tkzDrawSegments(a,b r,t b,c)
\tkzLabelSegment[sloped](a,b){ab = \tkzUseLua{r}}
\end{tikzpicture}
```



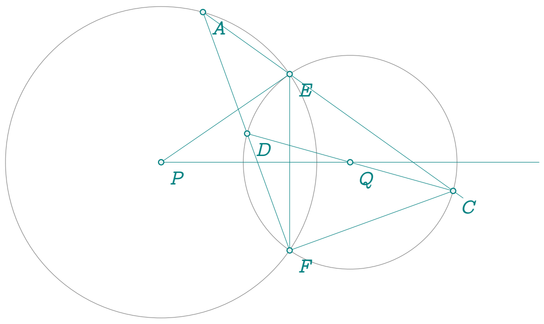
10.2 Methods of the class circle

Table 8: Circle methods.

Methods	Comments	
new(O,A)	`C.OA = circle : new (z.O,z.A)`	circle center O through A
radius(O,r)	`C.OA = circle : radius (z.O,2)`	circle center O radius =2 cm
diameter(A,B)	`C.OA = circle :diameter(z.A,z.B)`	circle diameter $[AB]$
Points		
antipode (pt)	`z.C = C.OA: antipode (z.B)`	$[BC]$ is a diameter
inversion (pt)	`z.Bp = C.AC: inversion (z.B)`	
midarc (pt,pt)	`z.D = C.AB: midarc (z.B,z.C)`	D is the midarc of $\widearc{BC}$
point (r)	`z.E = C.AB: point (0.25)`	r between 0 and 1
random_pt(lower, upper)		
internal_similitude (C)	`z.I  = C.one : internal_similitude (C.two)`	
external_similitude (C)	`z.J  = C.one : external_similitude (C.two)`	
radical_center (C1<,C2>)	or only (C1)	Refer to 23.30
Lines		
radical_axis (C)	Refer to (23.2 ; 23.26 ; 23.27 ; 23.28 ; 23.29)	
tangent_at (pt)	`z.P = C.OA: tangent_at (z.M)`	Refer to (10.2.2 ; 9.2.11)
tangent_from (pt)	`z.M,z.N = C.OA: tangent_from (z.P)`	Refer to ((iii))
inversion (line)	`L or C = C.AC: inversion (L.EF)`	Refer to (10.2.5)
common_tangent (C)	`z.a,z.b = C.AC: common_tangent (C.EF)`	Refer to (10.4 ; 10.5)
Circles		
orthogonal_from (pt)	`C = C.OA: orthogonal_from (z.P)`	Refer to (10.2.1 ; 10.5 ; 23.36 ; 23.40)
orthogonal_through (pta,ptb)	`C = C.OA: orthogonal_through (z.z1,z.z2)`	Refer to (23.10)
inversion (...)	`C.AC: inversion (pt, pts, L or C )`	Refer to 10.2.3, 10.2.4, 10.2.5, 10.2.6
midcircle (C)	`C.inv = C.OA: midcircle (C.EF)`	Refer to 10.2.7
radical_circle (C1<,C2>)	or only (C1)	Refer to 23.31
Miscellaneous		
power (pt)	`p = C.OA: power (z.M)`	Refer to (23.42 ; 23.43 ; 23.34)
in_out (pt)	`C.OA : in_out (z.M)`	Refer to (10.6)
in_out_disk (pt)	`C.OA : in_out_disk (z.M)`	Refer to (10.6)
draw ()	for further use	
circles_position (C1)	result = string	Refer to (10.3)

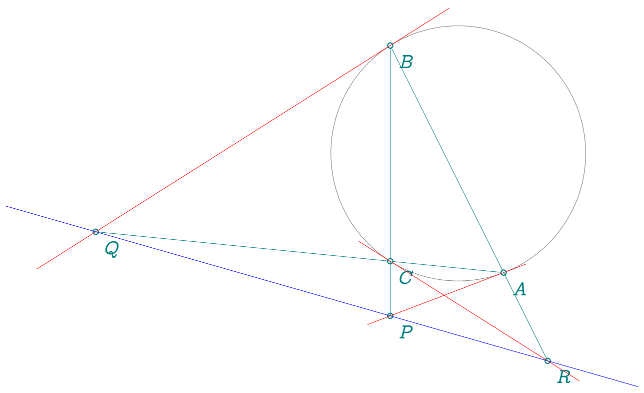
10.2.1 Altshiller

```
\begin{tkzelements}
   z.P   = point : new (0,0)
   z.Q   = point : new (5,0)
   z.I   = point : new (3,2)
   C.QI  = circle :    new (z.Q,z.I)
   C.PE  = C.QI : orthogonal_from (z.P)
   z.E   = C.PE.through
   C.QE  = circle :    new (z.Q,z.E)
   _,z.F = intersection (C.PE,C.QE)
   z.A   = C.PE: point (1/9)
   L.AE  = line : new (z.A,z.E)
   _,z.C = intersection (L.AE,C.QE)
   L.AF  = line : new (z.A,z.F)
   L.CQ  = line : new (z.C,z.Q)
   z.D   = intersection (L.AF,L.CQ)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(P,E Q,E)
   \tkzDrawLines[add=0 and 1](P,Q)
   \tkzDrawLines[add=0 and 2](A,E)
   \tkzDrawSegments(P,E E,F F,C A,F C,D)
   \tkzDrawPoints(P,Q,E,F,A,C,D)
   \tkzLabelPoints(P,Q,E,F,A,C,D)
\end{tikzpicture}
```



10.2.2 Lemoine

```
\begin{tkzelements}
   scale = 1.6
   z.A   = point: new (1,0)
   z.B   = point: new (5,2)
   z.C   = point: new (1.2,2)
   T     = triangle: new(z.A,z.B,z.C)
   z.O   = T.circumcenter
   C.OA  = circle: new (z.O,z.A)
   L.tA  = C.OA: tangent_at (z.A)
   L.tB  = C.OA: tangent_at (z.B)
   L.tC  = C.OA: tangent_at (z.C)
   z.P   = intersection (L.tA,T.bc)
   z.Q   = intersection (L.tB,T.ca)
   z.R   = intersection (L.tC,T.ab)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon[teal](A,B,C)
   \tkzDrawCircle(O,A)
   \tkzDrawPoints(A,B,C,P,Q,R)
   \tkzLabelPoints(A,B,C,P,Q,R)
   \tkzDrawLine[blue](Q,R)
   \tkzDrawLines[red](A,P B,Q R,C)
   \tkzDrawSegments(A,R C,P C,Q)
\end{tikzpicture}
```



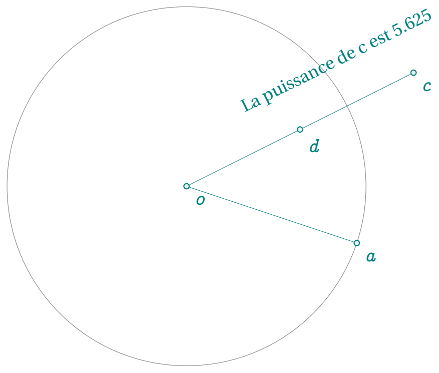
10.2.3 Inversion: point, line and circle

The `inversion` method can be used on a point, a line or a circle. Depending on the type of object, the function determines the correct algorithm to use.

10.2.4 Inversion: point

The `inversion` method can be used on a point, a group of points, a line or a circle. Depending on the type of object, the function determines the correct algorithm to use.

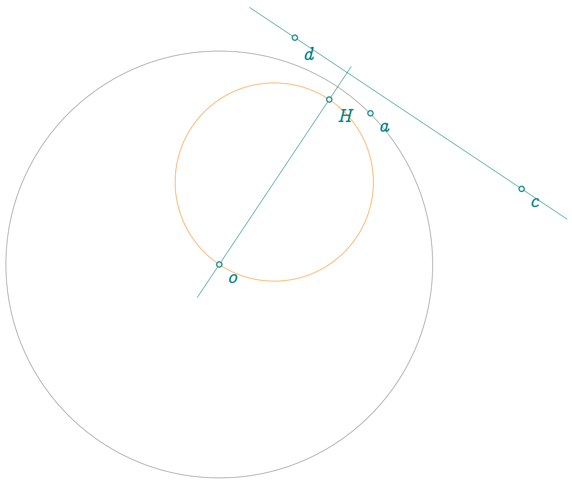
```
\begin{tkzelements}
   z.o   = point:    new (-1,2)
   z.a   = point:    new (2,1)
   C.oa  = circle:   new (z.o,z.a)
   z.c   = point:    new (3,4)
   z.d   = C.oa:     inversion (z.c)
   p     = C.oa:     power (z.c)
\end{tkzelements}
\begin{tikzpicture}
    \tkzGetNodes
    \tkzDrawCircle(o,a)
    \tkzDrawSegments(o,a o,c)
    \tkzDrawPoints(a,o,c,d)
    \tkzLabelPoints(a,o,c,d)
    \tkzLabelSegment[sloped,above=1em](c,d){%
    Power of c with respect to C is \tkzUseLua{p}}
 \end{tikzpicture}
```



10.2.5 Inversion: line

The result is either a straight line or a circle.

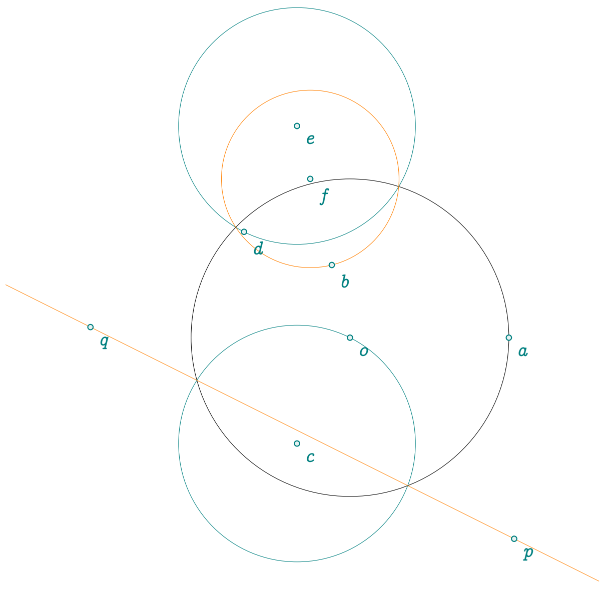
```
\begin{tkzelements}
   z.o       = point:    new (-1,1)
   z.a       = point:    new (1,3)
   C.oa      = circle:   new (z.o,z.a)
   z.c       = point:    new (3,2)
   z.d       = point:    new (0,4)
   L.cd      = line:    new (z.c,z.d)
   C.OH      = C.oa: inversion (L.cd)
   z.O,z.H   = get_points(C.OH)
\end{tkzelements}
\begin{tikzpicture}
    \tkzGetNodes
    \tkzDrawCircles(o,a O,H)
    \tkzDrawLines(c,d o,H)
    \tkzDrawPoints(a,o,c,d,H)
    \tkzLabelPoints(a,o,c,d,H)
 \end{tikzpicture}
```



10.2.6 Inversion: circle

The result is either a straight line or a circle.

```
\begin{tkzelements}
scale = .7
z.o,z.a  = point:  new (-1,3),point:  new (2,3)
z.c      = point:  new (-2,1)
z.e,z.d  = point:  new (-2,7),point: new (-3,5)
C.oa     = circle: new (z.o,z.a)
C.ed     = circle: new (z.e,z.d)
C.co     = circle: new (z.c,z.o)
obj      = C.oa: inversion (C.co)
   if obj.type == "line"
   then z.p,z.q = get_points(obj)
   else z.f,z.b = get_points(obj) end
obj      = C.oa: inversion(C.ed)
if obj.type == "line"
then z.p,z.q = get_points(obj)
else z.f,z.b = get_points(obj) end
color = "orange"
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawCircles[black](o,a)
\tkzDrawCircles[teal](c,o e,d)
\tkzDrawCircles[\tkzUseLua{color}](f,b)
\tkzDrawLines[\tkzUseLua{color}](p,q)
\tkzDrawPoints(a,...,f,o,p,q)
\tkzLabelPoints(a,...,f,o,p,q)
\end{tikzpicture}
```



10.2.7 midcircle

From Eric Danneels and Floor van Lamoen: A midcircle of two given circles is a circle that swaps the two given circles by inversion. Midcircles are in the same pencil of circles as the given circles. The center of the midcircle(s) is one or both of the centers of similitude. We can distinguish four cases:

(i) The two given circles intersect: there are two midcircles with centers at the centers of similitude of the given circles;

(ii) One given circle is in the interior of the other given circle. Then there is one midcircle with center of similitude at the internal center of similitude of the given circles;

(iii) One given circle is in the exterior of the other given circle. Then there is one midcircle with center at the external center of similitude of the given circles. Clearly the tangency cases can be seen as limit cases of the above;

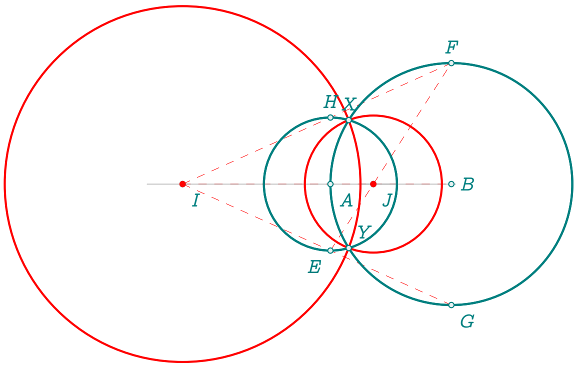
(iv) If the circles intersect in a single point, the unique midcircle has center at the external similitude center or at internal similitude center.

Let's look at each of these cases:

(i) If the two given circles intersect, then there are two circles of inversion through their common points, with centers at the centers of similitudes. The two midcircles bisect their angles and are orthogonal to each other. The centers of the midcircles are the internal center of similitude and the external center of similitude I and J.

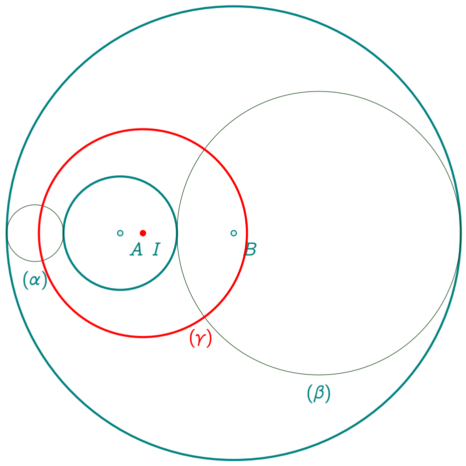
Consider two intersecting circles $(\mathcal{A})$ and $(\mathcal{B})$. We can obtain the centers of similarity of these two circles by constructing EH and FG two diameters parallel of the circles $(\mathcal{A})$ and $(\mathcal{B})$. The line (GE) intercepts the line (AB) in J and the line (EF) intercepts the line (AB) in I. The circles $(\mathcal{I})$ and $(\mathcal{J})$ are orthogonal and are the midcircles of $(\mathcal{A})$ and $(\mathcal{B})$. The division $(A,B;I,J)$ is harmonic.

```
\begin{tkzelements}
scale = .8
z.A = point : new ( 1 , 0 )
z.B = point : new ( 3 , 0 )
z.O = point : new ( 2.1, 0 )
z.P = point : new ( 1 ,0 )
C.AO = circle : new (z.A,z.O)
C.BP = circle : new (z.B,z.P)
z.E = C.AO.south
z.H = C.AO.north
z.F = C.BP.north
z.G = C.BP.south
C.IT,C.JV = C.AO : midcircle (C.BP)
z.I,z.T = get_points (    C.IT    )
z.J,z.V = get_points (    C.JV    )
z.X,z.Y = intersection (C.AO,C.BP)
\end{tkzelements}
```



(ii) One given circle is in the interior of the other given circle.

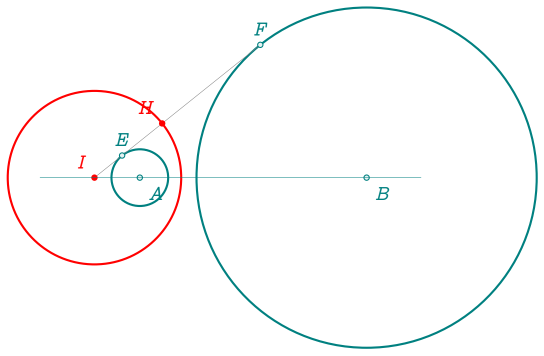
```
\begin{tkzelements}
   scale =.75
z.A = point : new ( 3 , 0 )
z.B = point : new ( 5 , 0 )
z.O = point : new ( 2 , 0 )
z.P = point : new ( 1 , 0 )
L.AB = line :  new (z.A,z.B)
C.AO = circle : new (z.A,z.O)
C.BP = circle : new (z.B,z.P)
z.R,z.S = intersection (L.AB,C.BP)
z.U,z.V = intersection (L.AB,C.AO)
C.SV = circle   :  diameter (z.S,z.V)
C.UR = circle   :  diameter (z.U,z.R)
z.x = C.SV.center
z.y = C.UR.center
C.IT = C.AO : midcircle (C.BP)
z.I,z.T = get_points (    C.IT    )
\end{tkzelements}
```



This case is a little more complicated. We'll construct the two circles (α) and (β) tangent to the two given circles. Then we construct the radical circle orthogonal to the circles (α) and (β). Its center is the radical center as well as the center of internal similtude of circles of center A and B.

(iii) When the two given circles are external to each other, we construct the external center of similitude of the two given circles. I is the center of external similarity of the two given circles. To obtain the inversion circle, simply note that H is such that $IH^2 = IE \times IF$.

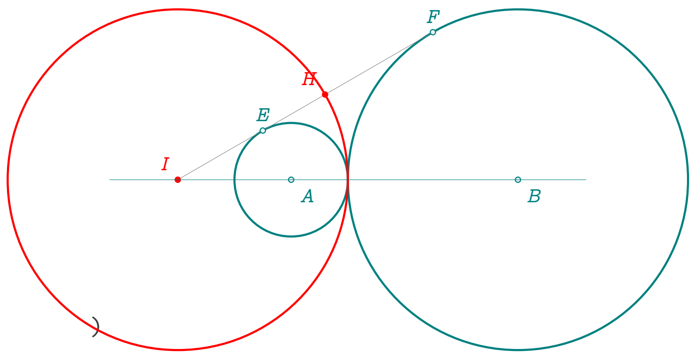
```
\begin{tkzelements}
scale=.75
local a,b,c,d
z.A = point : new ( 0 , 0 )
z.B = point : new ( 4 , 0 )
z.a = point :  new ( .5 , 0)
z.b = point :  new ( 1 , 0)
C.Aa = circle  :  new (z.A,z.a)
C.Bb = circle  :  new (z.B,z.b)
L.AB = line :  new (z.A,z.B)
z.E = C.Aa.north
z.F = C.Bb.north
L.EF = line :  new (z.E,z.F)
C.IT =  C.Aa : midcircle (C.Bb)
z.I,z.T = get_points (   C.IT   )
L.TF = C.Bb : tangent_from (z.I)
z.H = intersection (L.TF,C.IT)
z.E = intersection (L.TF,C.Aa)
z.F=L.TF.pb
\end{tkzelements}
```



(iv) Consider two tangent circles $(\mathcal{A})$ and $(\mathcal{B})$,

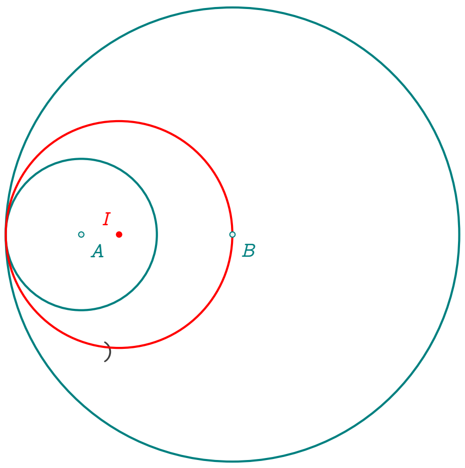
– $(\mathcal{B})$ being external and angent to $(\mathcal{A})$. The construction is identical to the previous one.

```
\begin{tkzelements}
scale=.75
local a,b,c,d
z.A = point : new ( 0 , 0 )
z.B = point : new ( 4 , 0 )
z.a = point :  new ( 1 , 0)
z.b = point :  new ( 1 , 0)
C.Aa = circle  :  new (z.A,z.a)
C.Bb = circle  :  new (z.B,z.b)
L.AB = line :  new (z.A,z.B)
z.E = C.Aa.north
z.F = C.Bb.north
L.EF = line :  new (z.E,z.F)
C.IT =  C.Aa : midcircle (C.Bb)
z.I,z.T = get_points (        C.IT        )
L.TF = C.Bb : tangent_from (z.I)
z.H = intersection (L.TF,C.IT)
z.E = intersection (L.TF,C.Aa)
z.F=L.TF.pb
\end{tkzelements}
```



– When one of the given circles is inside and tangent to the other, the construction is easy.

```
\begin{tkzelements}
z.A      = point : new ( 2 , 0 )
z.B      = point : new ( 4 , 0 )
z.a      = point :  new ( 1 , 0)
z.b      = point :  new ( 1 , 0)
C.Aa     = circle  :  new (z.A,z.a)
C.Bb     = circle  :  new (z.B,z.b)
C.IT     =  C.Aa : midcircle (C.Bb)
z.I,z.T  = get_points (        C.IT        )
\end{tkzelements}
```

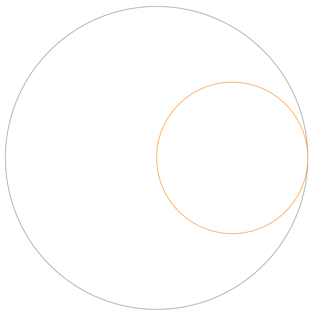


10.3 Circles_position

This function returns a string indicating the position of the circle in relation to another. Useful for creating a function. Cases are:

- outside
- outside tangent
- inside tangent
- inside
- intersect

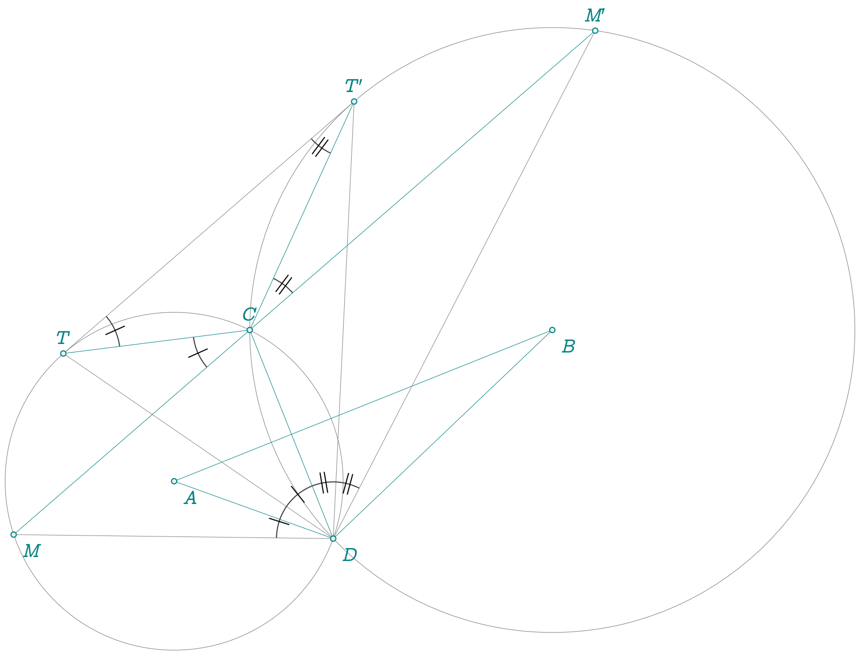
```
\begin{tkzelements}
   z.A      = point : new ( 0   , 0   )
   z.a      = point : new ( 3   , 0   )
   z.B      = point : new ( 2   , 0   )
   z.b      = point : new ( 3   , 0   )
   C.Aa     = circle: new (z.A,z.a)
   C.Bb     = circle: new (z.B,z.b)
   position = C.Aa : circles_position (C.Bb)
   if position == "inside tangent"
   then color = "orange"
   else color = "blue" end
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(A,a)
   \tkzDrawCircle[color=\tkzUseLua{color}](B,b)
\end{tikzpicture}
```



10.4 Common tangent: Angle of two intersecting circles

Let be a tangent common to both circles at T and T' (closest to C). Let a secant parallel to this tangent pass through C. Then the segment $[TT']$ is seen from the other common point D at an angle equal to half the angle of the two circles.

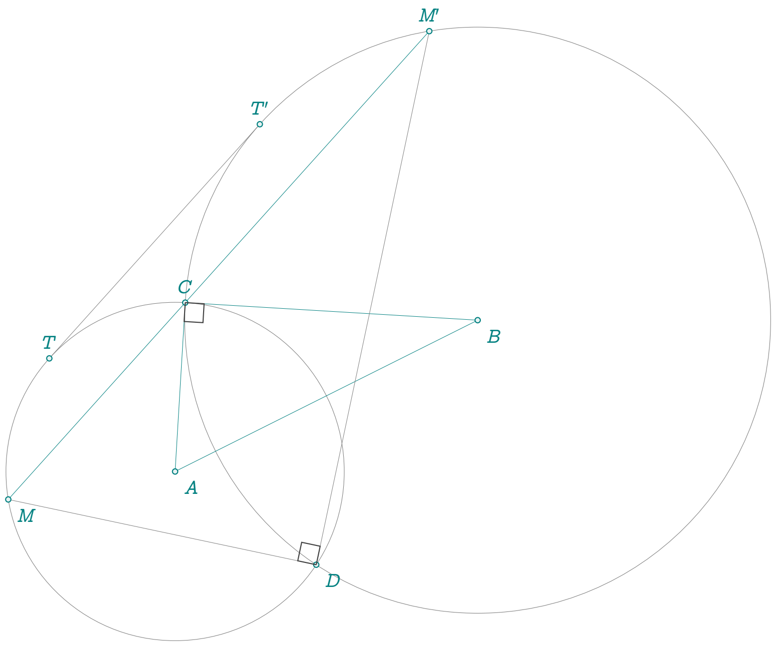
```
\begin{tkzelements}
   z.A    = point : new ( 0   , 0   )
   z.B   = point : new ( 5  , 2  )
   L.AB = line : new ( z.A , z.B )
   z.C   = point : new ( 1 , 2 )
   C.AC  = circle : new (z.A,z.C)
   C.BC  = circle : new (z.B,z.C)
   z.T,z.Tp = C.AC : common_tangent (C.BC)
   L.TTp = line : new (z.T,z.Tp)
   z.M   = C.AC : point (0.45)
   L.MC  =line : new (z.M,z.C)
   z.Mp  = intersection (L.MC, C.BC)
   L.mm = L.TTp : ll_from (z.C)
   _,z.M = intersection (L.mm, C.AC)
   z.Mp = intersection (L.mm, C.BC)
   _,z.D = intersection (C.AC,C.BC)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,C B,C)
   \tkzDrawSegments(M,M' A,D B,D A,B C,D T,C T',C)
   \tkzDrawSegments[gray](D,M D,M' T,T' D,T D,T')
   \tkzDrawPoints(A,B,C,D,M,M',T,T')
   \tkzLabelPoints(A,B,D,M)
   \tkzLabelPoints[above](C,M',T,T')
   \tkzMarkAngles[mark=|,size=.75](T,C,M C,T,T' C,D,T T,D,M)
   \tkzMarkAngles[mark=||,size=.75](M',C,T' T,T',C T',D,C M',D,T')
\end{tikzpicture}
```



10.5 Common tangent: orthogonality

For two circles to be orthogonal, it is necessary and sufficient for a secant passing through one of their common points to be seen from the other common point at a right angle.

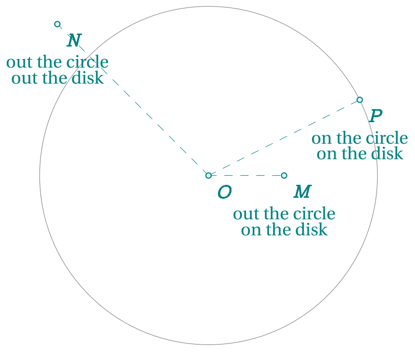
```
\begin{tkzelements}
   z.A   = point : new ( 0  , 0  )
   z.B   = point : new ( 4  , 2  )
   L.AB = line : new ( z.A , z.B )
   z.a   = point : new ( 1 , 2 )
   C.Aa  = circle : new (z.A,z.a)
   C.BC = C.Aa : orthogonal_from (z.B)
   z.C,z.D = intersection (C.Aa,C.BC)
   C.AC  = circle : new (z.A,z.C)
   z.T,z.Tp = C.AC : common_tangent (C.BC)
   L.TTp = line : new (z.T,z.Tp)
   z.M   = C.AC : point (0.45)
   L.MC  =line : new (z.M,z.C)
   z.Mp  = intersection (L.MC, C.BC)
   L.mm = L.TTp : ll_from (z.C)
   _,z.M = intersection (L.mm, C.AC)
   z.Mp = intersection (L.mm, C.BC)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,C B,C)
   \tkzDrawSegments(M,M' A,C B,C A,B)
   \tkzDrawSegments[gray](D,M D,M' T,T')
   \tkzDrawPoints(A,B,C,D,M,M',T,T')
   \tkzLabelPoints(A,B,D,M)
   \tkzLabelPoints[above](C,M',T,T')
   \tkzMarkRightAngles(M',D,M A,C,B)
\end{tikzpicture}
```



10.6 In_out for circle and disk

```
\begin{tkzelements}
z.O = point : new (0,0)
z.A = point : new (1,2)
```

```
C.OA = circle : new (z.O,z.A)
z.N = point : new (-2,2)
z.M = point : new (1,0)
z.P = point : new (2,1)
BCm = C.OA : in_out (z.M)
BDm = C.OA : in_out_disk (z.M)
BCn = C.OA : in_out (z.N)
BDn = C.OA : in_out_disk (z.N)
BCp = C.OA : in_out (z.P)
BDp = C.OA : in_out_disk (z.P)
\end{tkzelements}
\def\tkzPosPoint#1#2#3#4{%
\tkzLabelPoints(O,M,N,P)
   \ifthenelse{\equal{\tkzUseLua{#1}}{true}}{
   \tkzLabelPoint[below=#4pt,font=\scriptsize](#2){on  the #3}}{%
   \tkzLabelPoint[below=#4pt,font=\scriptsize](#2){out  the #3}}}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawSegments[dashed](O,M O,N O,P)
\tkzDrawCircle(O,A)
\tkzDrawPoints(O,M,N,P)
\tkzPosPoint{BCm}{M}{circle}{8}
\tkzPosPoint{BCn}{N}{circle}{8}
\tkzPosPoint{BCp}{P}{circle}{8}
\tkzPosPoint{BDm}{M}{disk}{14}
\tkzPosPoint{BDn}{N}{disk}{14}
\tkzPosPoint{BDp}{P}{disk}{14}
\end{tikzpicture}
```



11 Class triangle

11.1 Attributes of a triangle

The triangle object is created using the `new` method, for example with

```
Creation T.ABC = triangle : new ( z.A , z.B , z.C )
```

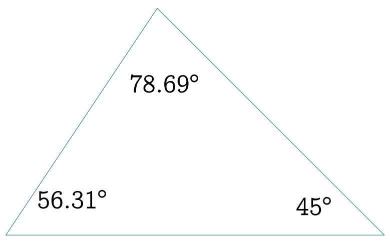
(Refer to examples: 23.3; 23.4; 23.9). Multiple attributes are then created.

Table 9: Triangle attributes.

Attributes	Application
pa	T.ABC.pa
pb	T.ABC.pb
pc	T.ABC.pc
type	'triangle'
circumcenter	T.ABC.circumcenter
centroid	T.ABC.centroid
incenter	T.ABC.incenter
orthocenter	T.ABC.orthocenter
eulercenter	T.ABC.eulercenter
spiekercenter	T.ABC.spiekercenter
a	It's the length of the side opposite the first vertex
b	It's the length of the side opposite the second verte
c	It's the length of the side opposite the third vertex
alpha	Vertex angle of the first vertex
beta	Vertex angle of the second vertex
gamma	Vertex angle of the third vertex
ab	Line defined by the first two points of the triangle
bc	Line defined by the last two points
ca	Line defined by the last and the first points of the triangle

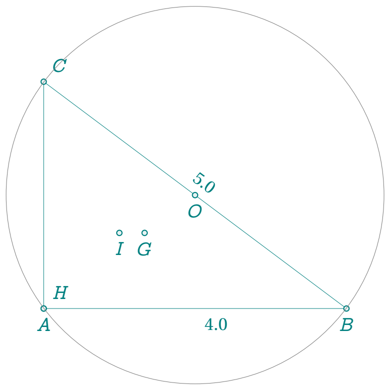
11.2 Triangle attributes: angles

```
\begin{tkzelements}
  z.A      = point: new(0,0)
  z.B      = point: new(5,0)
  z.C      = point: new(2,3)
  T.ABC    = triangle: new (z.A,z.B,z.C)
\end{tkzelements}
\def\wangle#1{\tkzDN[2]{%
  \tkzUseLua{math.deg(T.ABC.#1)}}}
\begin{tikzpicture}
\tkzGetNodes
  \tkzDrawPolygons(A,B,C)
  \tkzLabelAngle(B,A,C){$\wangle{alpha}^\circ$}
  \tkzLabelAngle(C,B,A){$\wangle{beta}^\circ$}
  \tkzLabelAngle(A,C,B){$\wangle{gamma}^\circ$}
\end{tikzpicture}
```



11.2.1 Example: triangle attributes

```
\begin{tkzelements}
   z.A   = point: new (0 , 0)
   z.B   = point: new (4 , 0)
   z.C   = point: new (0 , 3)
   T.ABC = triangle : new (z.A,z.B,z.C)
   z.O   = T.ABC.circumcenter
   z.I   = T.ABC.incenter
   z.H   = T.ABC.orthocenter
   z.G   = T.ABC.centroid
   a     = T.ABC.a
   b     = T.ABC.b
   c     = T.ABC.c
   alpha = T.ABC.alpha
   beta  = T.ABC.beta
   gamma = T.ABC.gamma
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawPoints(A,B,C,O,G,I,H)
   \tkzLabelPoints[below](A,B,O,G,I)
   \tkzLabelPoints[above right](H,C)
   \tkzDrawCircles(O,A)
   \tkzLabelSegment[sloped](A,B){\tkzUseLua{c}}
   \tkzLabelSegment[sloped,above](B,C){\tkzUseLua{a}}
\end{tikzpicture}
```



11.3 Methods of the class triangle

Table 10: triangle methods.

Methods	Comments
`new` (a, b ,c)	`T.ABC = triangle : new (z.A,z.B,z.C)`
...	T or `T.name` with what you want for name, is possible.
Points	
`lemoine_point ()`	`T.ABC : lemoine_point ()` intersection os the symmedians
`symmedian_point ()`	Lemoine point or the Grebe point
`bevan_point ()`	Circumcenter of the excentral triangle
`mittenpunkt_point ()`	Symmedian point of the excentral triangle
`gergonne_point ()`	Intersection of the three cevians that lead to the contact points
`nagel_point ()`	Intersection of the three cevians that lead to the extouch points
`feuerbach_point ()`	The point at which the incircle and euler circle are tangent.
`spieker_center ()`	Incenter of the medial triangle
`barycenter (ka,kb,kc)`	`T.ABC: barycenter (2,1,1)` barycenter of `({A,2},{B,1},{C,1})`
`base (u,v)`	`z.D = T.ABC: base(1,1)` –> ABDC is a parallelogram
`projection (p)`	Projection of a point on the sides
`euler_points ()`	Euler points of euler circle
`nine_points ()`	9 Points of the euler circle
`parallelogram ()`	`z.D = T.ABC : parallelogram ()` –> ABCD is a parallelogram
Lines	
`altitude (n)`	`L.AHa = T.ABC : altitude ()` n empty or 0 line from A [a]
`bisector (n)`	`L.Bb = T.ABC : bisector (1)` n = 1 line from B [b]
`bisector_ext(n)`	n=2 line from the third vertex.
`symmedian_line (n)`	Cevian with respect to Lemoine point.
`euler_line ()`	the line through N ,G, H and O if the triangle is not equilateral [c]
`antiparallel(pt,n)`	n=0 antiparallel through pt to (BC), n=1 to (AC) etc.
Circles	
`euler_circle ()`	`C.NP = T.ABC : euler_circle ()` –> N euler point [d]
`circum_circle ()`	`C.OA = T.ABC : circum ()` Triangle's circumscribed circle
`in_circle ()`	Inscribed circle of the triangle
`ex_circle (n)`	Circle tangent to the three sides of the triangle ; n =1 swap ; n=2 2 swap
`first_lemoine_circle ()`	The center is the midpoint between Lemoine point and the circumcenter.[e]
`second_lemoine_circle ()`	Refer to example 23.58
`spieker_circle ()`	The incircle of the medial triangle

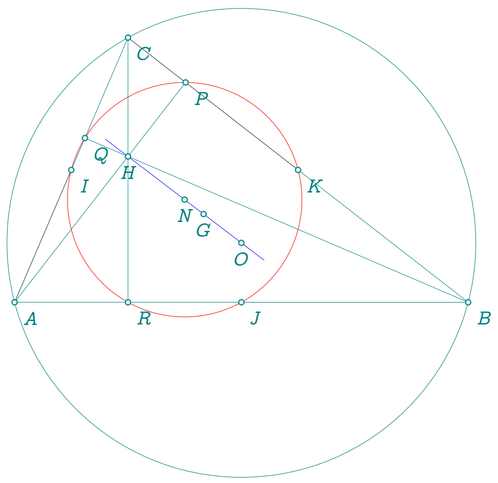
a `z.Ha = L.AHa.pb` recovers the common point of the opposite side and altitude. The method `orthic` is usefull.
b `_,z.b = get_points(L.Bb)` recovers the common point of the opposite side and bisector.
c N center of nine points circle, G centroid, H orthocenter , O circum center
d The midpoint of each side of the triangle, the foot of each altitude, the midpoint of the line segment from each vertex of the triangle to the orthocenter.
e Through the Lemoine point draw lines parallel to the triangle's sides. The points where the parallel lines intersect the sides of ABC then lie on a circle known as the first Lemoine circle.

Remark: If you don't need to use the triangle object several times, you can obtain a bisector or a altitude with the next functions
`bisector (z.A,z.B,z.C)` and `altitude (z.A,z.B,z.C)` Refer to (28)

Methods	Comments
Triangles	
`orthic ()`	`T = T.ABC : orthic ()` triangle joining the feet of the altitudes
`medial ()`	`T = T.ABC : medial ()` triangle with vertices at the midpoints
`incentral ()`	Cevian triangle of the triangle with respect to its incenter
`excentral ()`	Triangle with vertices corresponding to the excenters
`extouch ()`	Triangle formed by the points of tangency with the excircles
`intouch ()`	Contact triangle formed by the points of tangency of the incircle
`tangential ()`	Triangle formed by the lines tangent to the circumcircle at the vertices
`feuerbach ()`	Triangle formed by the points of tangency of the euler circle with the excircles
`anti ()`	Anticomplementary Triangle The given triangle is its medial triangle.
`cevian (pt)`	Triangle formed with the endpoints of the three cevians with respect to `pt`.
`symmedian ()`	Triangle formed with the intersection points of the symmedians.
`euler ()`	Triangle formed with the euler points
Ellipses	
`steiner_inellipse ()`	Refer to ex. (11.4.1)
`steiner_circumellipse ()`	Refer to ex. (11.4.1)
`euler_ellipse ()`	Refer to ex. (11.4)
Miscellaneous	
`area ()`	$\mathcal{A}$ `= T.ABC: area ()`
`barycentric_coordinates (pt)`	Triples of numbers corresponding to masses placed at the vertices
`in_out (pt)`	Boolean. Test if `pt` is inside the triangle
`check_equilateral ()`	Boolean. Test if the triangle is equilateral

11.3.1 Euler line

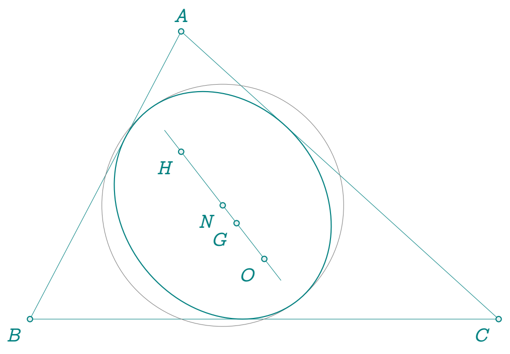
```
\begin{tkzelements}
   z.A          = point: new (0 , 0)
   z.B          = point: new (6 , 0)
   z.C          = point: new (1.5 , 3.5)
   T.ABC        = triangle: new (z.A,z.B,z.C)
   z.O          = T.ABC.circumcenter
   z.G          = T.ABC.centroid
   z.N          = T.ABC.eulercenter
   z.H          = T.ABC.orthocenter
   z.P,z.Q,z.R  = get_points (T.ABC: orthic())
   z.K,z.I,z.J  = get_points (T.ABC: medial ())
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines[blue](O,H)
   \tkzDrawCircle[red](N,I)
   \tkzDrawCircles[teal](O,A)
   \tkzDrawSegments(A,P B,Q C,R)
   \tkzDrawSegments[red](A,I B,J C,K)
   \tkzDrawPolygons(A,B,C)
   \tkzDrawPoints(A,B,C,N,I,J,K,O,P,Q,R,H,G)
   \tkzLabelPoints(A,B,C,I,J,K,P,Q,R,H)
   \tkzLabelPoints[below](N,O,G)
\end{tikzpicture}
```



11.4 Euler ellipse

Example of obtaining the Euler circle as well as the Euler ellipse.

```
\begin{tkzelements}
z.A     = point: new (2,3.8)
z.B     = point: new (0 ,0)
z.C     = point: new (6.2 ,0)
L.AB    = line : new ( z.A , z.B )
T.ABC   = triangle: new (z.A,z.B,z.C)
z.K     = midpoint (z.B,z.C)
E.euler = T.ABC : euler_ellipse ()
z.N     = T.ABC.eulercenter
C.euler = circle : new (z.N,z.K)
ang     = math.deg(E.euler.slope)
z.O     = T.ABC.circumcenter
z.G     = T.ABC.centroid
z.H     = T.ABC.orthocenter
\end{tkzelements}
```

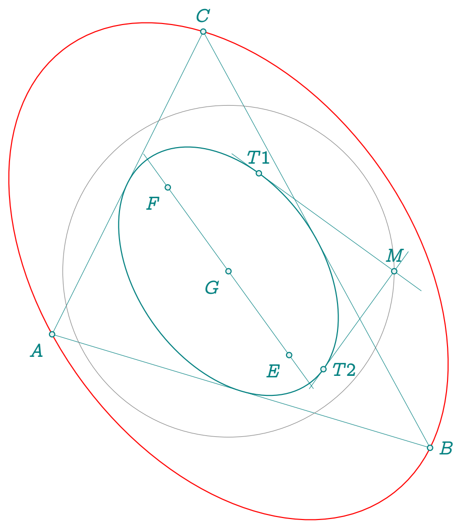


```
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C)
\tkzDrawCircle(N,K)
\tkzDrawEllipse[teal](N,\tkzUseLua{E.euler.Rx},
      \tkzUseLua{E.euler.Ry},\tkzUseLua{ang})
\tkzDrawLine(O,H)
\tkzDrawPoints(A,B,C,N,O,H,G)
\tkzLabelPoints[below left](B,C,N,O,H,G)
\tkzLabelPoints[above](A)
\end{tikzpicture}
```

11.4.1 Steiner inellipse and circumellipse

In this example, the inner and outer Steiner ellipses, referred to as the "inellipse" and "circumellipse" (Mathworld.com), respectively, along with the orthoptic circle, are depicted.. The triangle must be acutangle.

```
\begin{tkzelements}
  scale     = .5
  z.A       = point: new (1 , 4)
  z.B       = point: new (11 , 1)
  z.C       = point: new (5 , 12)
  T.ABC     = triangle: new(z.A,z.B,z.C)
  E         = T.ABC: steiner_inellipse ()
  z.G       = E.center
  ang       = math.deg(E.slope)
  z.F       = E.Fa
  z.E       = E.Fb
  C         = E: orthoptic_circle ()
  z.w       = C.center
  z.o       = C.through
  EE        = T.ABC : steiner_circumellipse ()
  z.M       = C : point (0)
  L.T1,L.T2= E : tangent_from (z.M)
  z.T1      = L.T1.pb
  z.T2      = L.T2.pb
\end{tkzelements}
```

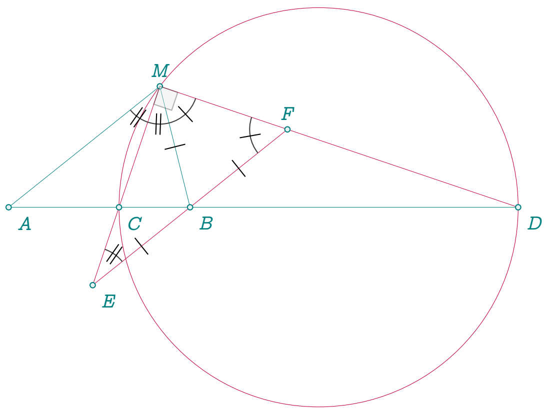


```
\begin{tikzpicture}
\tkzGetNodes
```

```
\tkzDrawPolygon(A,B,C)
\tkzDrawCircles(w,o)
\tkzDrawEllipse[teal](G,\tkzUseLua{E.Rx},
  \tkzUseLua{E.Ry},\tkzUseLua{ang})
\tkzDrawEllipse[red](G,\tkzUseLua{EE.Rx},
  \tkzUseLua{EE.Ry},\tkzUseLua{ang})
\tkzDrawLines(F,E M,T1 M,T2) %
\tkzDrawPoints(A,B,C,F,E,G,M,T1,T2)
\tkzLabelPoints[above](C,M,T1)
\tkzLabelPoints[right](T2,B)
\tkzLabelPoints[below left](A,F,E,G)
\end{tikzpicture}
```

11.5 Harmonic division and bisector

```
\begin{tkzelements}
   scale    =  .4
   z.A      = point: new (0 , 0)
   z.B      = point: new (6 , 0)
   z.M      = point: new (5 , 4)
   T.AMB    = triangle : new (z.A,z.M,z.B)
   L.AB     = T.AMB.ca
   L.bis    = T.AMB : bisector (1)
   z.C      = L.bis.pb
   L.bisext = T.AMB : bisector_ext (1)
   z.D      = intersection (L.bisext,L.AB)
   L.CD     = line: new (z.C,z.D)
   z.O      = L.CD.mid
   L.AM     = line: new (z.A,z.M)
   L.LL     = L.AM : ll_from (z.B)
   L.MC     = line: new (z.M,z.C)
   L.MD     = line: new (z.M,z.D)
   z.E      = intersection (L.LL,L.MC)
   z.F      = intersection (L.LL,L.MD)
\end{tkzelements}
```



```
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,M)
   \tkzDrawCircle[purple](O,C)
   \tkzDrawSegments[purple](M,E M,D E,F)
   \tkzDrawSegments(D,B)
   \tkzDrawPoints(A,B,M,C,D,E,F)
   \tkzLabelPoints[below right](A,B,C,D,E)
   \tkzLabelPoints[above](M,F)
   \tkzMarkRightAngle[opacity=.4,fill=gray!20](C,M,D)
   \tkzMarkAngles[mark=||,size=.5](A,M,E E,M,B B,E,M)
   \tkzMarkAngles[mark=|,size=.5](B,M,F M,F,B)
   \tkzMarkSegments(B,E B,M B,F)
\end{tikzpicture}
```

12 Class ellipse

12.1 Attributes of an ellipse

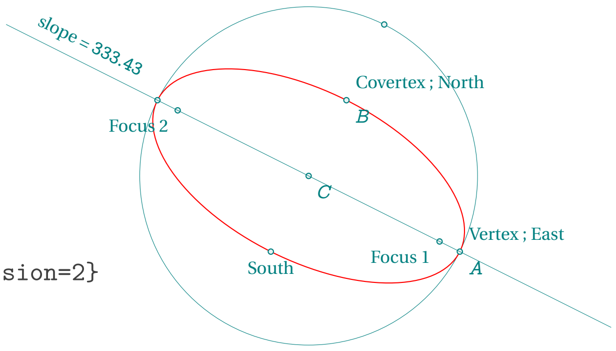
The first attributes are the three points that define the ellipse: : the `center` , the `vertex` and the`covertex`. The first method to define an ellipse is to give its center, then the point named **`vertex`** which defines the major axis and finally the point named **`covertex`** which defines the minor axis.

Table 11: Ellipse attributes.

Attributes	**Application**
`center`	center of the ellipse
`vertex`	point of the major axis and of the ellipse
`covertex`	point of the minor axis and of the ellipse
`type`	The type is 'ellipse'
`Rx`	Radius from center to vertex
`Ry`	Radius from center to covertex
`slope`	Slope of the line passes through the foci
`Fa`	First focus
`Fb`	Second focus
`south`	See next example 12.1.1
`north`	
`west`	
`east`	

12.1.1 Atributes of an ellipse: example

```
\begin{tkzelements}
   z.C   = point: new (3 , 2)
   z.A   = point: new (5 , 1)
   L.CA  = line : new (z.C,z.A)
   z.b   = L.CA.north_pa
   L     = line : new (z.C,z.b)
   z.B   = L : point (0.5)
   E     = ellipse: new (z.C,z.A,z.B)
   a     = E.Rx
   b     = E.Ry
   z.F1  = E.Fa
   z.F2  = E.Fb
   slope = math.deg(E.slope)
   z.E   = E.east
   z.N   = E.north
   z.W   = E.west
   z.S   = E.south
   z.Co  = E.covertex
   z.Ve  = E.vertex
\end{tkzelements}
\begin{tikzpicture}
   \pgfkeys{/pgf/number format/.cd,fixed,precision=2}
   \tkzGetNodes
   \tkzDrawCircles[teal](C,A)
   \tkzDrawEllipse[red](C,\tkzUseLua{a},\tkzUseLua{b},
   \tkzUseLua{slope})
   \tkzDrawPoints(C,A,B,b,W,S,F1,F2)
   \tkzLabelPoints(C,A,B)
   \tkzDrawLine[add = .5 and .5](A,W)
   \tkzLabelSegment[pos=1.5,above,sloped](A,W){%
    slope = \pgfmathprintnumber{\tkzUseLua{slope}}}
   \tkzLabelPoint[below](S){South}
   \tkzLabelPoint[below left](F1){Focus 1}
   \tkzLabelPoint[below left](F2){Focus 2}
   \tkzLabelPoint[above right](Ve){Vertex ; East}
   \tkzLabelPoint[above right](Co){Covertex ; North}
\end{tikzpicture}
```



12.2 Methods of the class ellipse

Before reviewing the methods and functions related to ellipses, let's take a look at how you can draw ellipses with `tkz-elements`. The **`\tkzDrawEllipse`** macro requires 4 arguments: the center of the ellipse, the long radius (on the focus axis), the short radius and the angle formed by the focus axis. The last three arguments must be transferred from **`tkzelements`** to **`tikzpicture`**. To do this, you'll need to use a macro: **`\tkzUseLua`** defined in `tkz-elements`. Refer to 6.1.2 or 20.6 or next examples.

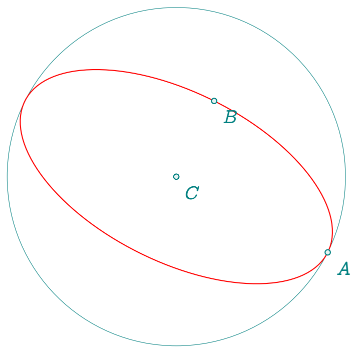
Table 12: Ellipse methods.

Methods	Example
`new (pc, pa ,pb)`	E = ellipse: new (center, vertex, covertex)
`foci (f1,f2,v)`	E = ellipse: foci (focus 1, focus 2, vertex)
`radii (c,a,b,sl)`	E = ellipse: radii (center, radius a, radius b, slope)
`in_out (pt)`	pt in/out of the ellipse
`tangent_at (pt)`	Refer to ex. 9.2.6
`tangent_from (pt)`	Refer to ex. 9.2.6
`point (t)`	vertex = point (0) covertex = point (0.25) ex Refer to 9.2.6
`orthoptic_circle ()`	Refer to ex. 11.4.1

12.2.1 Method new

The main method for creating a new ellipse is `new`. The arguments are three: `center`, `vertex` and `covertex` For attributes Refer to 12.

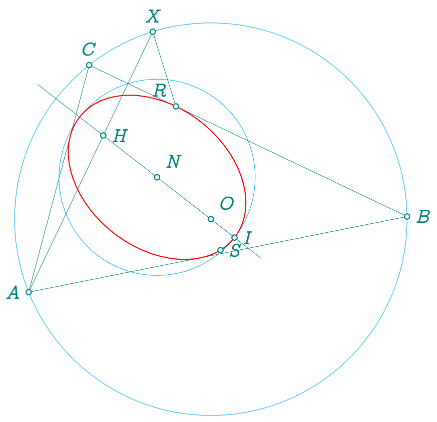
```
\begin{tkzelements}
   z.C      = point: new (3 , 2)
   z.A      = point: new (5 , 1)
   z.B      = z.C : homothety(0.5,
              z.C : rotation (math.pi/2,z.A))
   E        = ellipse: new (z.C,z.A,z.B)
   a        = E.Rx
   b        = E.Ry
   slope = math.deg(E.slope)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles[teal](C,A)
   \tkzDrawEllipse[red](C,\tkzUseLua{a},
         \tkzUseLua{b},\tkzUseLua{slope})
   \tkzDrawPoints(C,A,B)
   \tkzLabelPoints(C,A,B)
\end{tikzpicture}
```



The function `tkzUseLua (variable)` is used to transfer values to Ti*k*Z or `tkz-euclide`.

12.2.2 Method foci

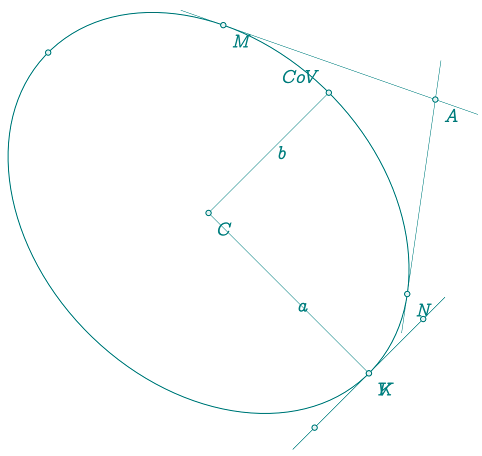
The first two points are the foci of the ellipse, and the third one is the vertex. We can deduce all the other characteristics from these points. *The function launches the* `new` *method, defining all the characteristics of the ellipse.*



```
\begin{tkzelements}
   z.A      = point: new (0 , 0)
   z.B      = point: new (5 , 1)
   L.AB     = line : new (z.A,z.B)
   z.C      = point: new (.8 , 3)
   T.ABC    = triangle: new (z.A,z.B,z.C)
   z.N      = T.ABC.eulercenter
   z.H      = T.ABC.orthocenter
   z.O      = T.ABC.circumcenter
   _,_,z.Mc = get_points (T.ABC: medial ())
   L.euler  = line: new (z.H,z.O)
   C.circum = circle: new (z.O,z.A)
   C.euler  = circle: new (z.N,z.Mc)
   z.i,z.j  = intersection (L.euler,C.circum)
   z.I,z.J  = intersection (L.euler,C.euler)
   E        = ellipse: foci (z.H,z.O,z.I)
   L.AH     = line: new (z.A,z.H)
   z.X      = intersection (L.AH,C.circum)
   L.XO     = line: new (z.X,z.O)
   z.R,z.S  = intersection (L.XO,E)
   a,b      = E.Rx,E.Ry
   ang      = math.deg(E.slope)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawCircles[cyan](O,A N,I)
   \tkzDrawSegments(X,R A,X)
   \tkzDrawEllipse[red](N,\tkzUseLua{a},
      \tkzUseLua{b},\tkzUseLua{ang})
   \tkzDrawLines[add=.2 and .5](I,H)
   \tkzDrawPoints(A,B,C,N,O,X,H,R,S,I)
   \tkzLabelPoints[above](C,X)
   \tkzLabelPoints[above right](N,O)
   \tkzLabelPoints[above left](R)
   \tkzLabelPoints[left](A)
   \tkzLabelPoints[right](B,I,S,H)
   \end{tikzpicture}
```

12.2.3 Method point and radii

The method `point` defines a point M of the ellipse whose coordinates are $(a \times cos(phi), b \times sin(phi))$. `phi` angle between (center,vertex) and (center,M)
The environment **`tkzelements`** *uses as* **`lua`** *the radian as unit for angles.*



```
\begin{tkzelements}
   z.C          = point: new (2 , 3)
   z.A          = point: new (6 , 5)
   a            = value(4)
   b            = value(3)
   ang          = math.deg(-math.pi/4)
   E            = ellipse: radii (z.C,a,b,-math.pi/4)
   z.V          = E : point (0)
   z.K          = E : point (1)
   z.CoV        = E : point (0.25)
   z.X          = E : point (0.5)
   L            = E :tangent_at (z.V)
   z.x,z.y      = get_points(L)
   L.ta,L.tb    = E :tangent_from (z.A)
   z.M          = L.ta.pb
   z.N          = L.tb.pb
   L.K          = E :tangent_at (z.K)
   z.ka,z.kb    = get_points(L.K)
\end{tkzelements}
```

```
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawSegments(C,V C,CoV)
   \tkzDrawLines(x,y A,M A,N ka,kb)
   \tkzLabelSegment(C,V){$a$}
   \tkzLabelSegment[right](C,CoV){$b$}
   \tkzDrawEllipse[teal](C,\tkzUseLua{a},\tkzUseLua{b},\tkzUseLua{ang})
   \tkzDrawPoints(C,V,CoV,X,x,y,M,N,A,K)
   \tkzLabelPoints(C,V,A,M,N,K)
   \tkzLabelPoints[above left](CoV)
\end{tikzpicture}
```

13 Class Quadrilateral

13.1 Quadrilateral Attributes

Points are created in the direct direction. A test is performed to check whether the points form a rectangle, otherwise compilation is blocked.

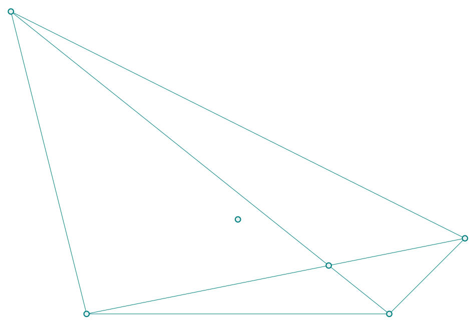
```
Creation  Q.new = rectangle : new (z.A,z.B,z.C,z.D)
```

Table 13: rectangle attributes.

Attributes	Application	
pa	z.A = Q.new.pa	
pb	z.B = Q.new.pb	
pc	z.C = Q.new.pc	
pd	z.D = Q.new.pd	
type	Q.new.type= 'quadrilateral'	
i	z.I = Q.new.i	intersection of diagonals
g	z.G = Q.new.g	barycenter
a	AB = Q.new.a	barycenter
b	BC = Q.new.b	barycenter
c	CD = Q.new.c	barycenter
d	DA = Q.new.d	barycenter
ab	Q.new.ab	line passing through two vertices
ac	Q.new.ca	idem.
ad	Q.new.ad	idem.
bc	Q.new.bc	idem.
bd	Q.new.bd	idem.
cd	Q.new.cd	idem.

13.1.1 Quadrilateral attributes

```
\begin{tkzelements}
z.A      = point : new ( 0 , 0 )
z.B      = point : new ( 4 , 0 )
z.C      = point : new ( 5 , 1 )
z.D      = point : new ( -1 , 4 )
Q.ABCD   = quadrilateral : new ( z.A , z.B , z.C , z.D )
z.I      = Q.ABCD.i
z.G      = Q.ABCD.g
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawSegments(A,C B,D)
\tkzDrawPoints(A,B,C,D,I,G)
\end{tikzpicture}
```



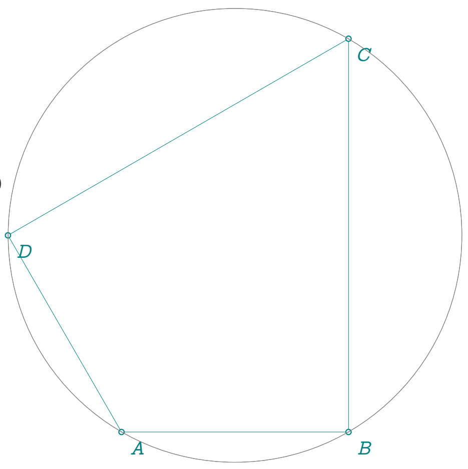
13.2 Quadrilateral methods

Table 14: Quadrilateral methods.

Methods	Comments
iscyclic ()	inscribed ? (Refer to next example)

13.2.1 Inscribed quadrilateral

```
\begin{tkzelements}
z.A      = point : new ( 0 , 0 )
z.B      = point : new ( 4 , 0 )
z.D      = point : polar ( 4 , 2*math.pi/3 )
L.DB     = line : new (z.D,z.B)
T.equ    = L.DB : equilateral ()
z.C      = T.equ.pc
Q.new    = quadrilateral : new (z.A,z.B,z.C,z.D)
bool     = Q.new : iscyclic ()
if bool == true then
C.cir    = triangle : new (z.A,z.B,z.C): circum_circle ()
z.O      = C.cir.center
end
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D)
\tkzLabelPoints(A,B,C,D)
\tkzDrawCircle(O,A)
\ifthenelse{\equal{\tkzUseLua{bool}}{true}}{
\tkzDrawCircle(O,A)}{}
\end{tikzpicture}
```



14 Class square

14.1 Square attributes

Points are created in the direct direction. A test is performed to check whether the points form a square. Otherwise, compilation is blocked."

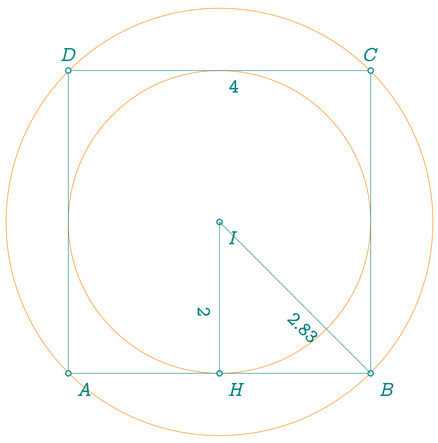
```
Creation S.AB = square : new (z.A,z.B,z.C,z.D)
```

Table 15: Square attributes.

Attributes	Application	
pa	z.A = S.AB.pa	
pb	z.B = S.AB.pb	
pc	z.C = S.AB.pc	
pd	z.D = S.AB.pd	
type	S.AB.type= 'square'	
side	s = S.AB.center	s = length of side
center	z.I = S.AB.center	center of the square
exradius	S.AB.exradius	radius of the circumscribed circle
inradius	S.AB.inxradius	radius of the inscribed circle
proj	S.AB.proj	projection of the center on one side
ab	S.AB.ab	line passing through two vertices
ac	S.AB.ca	idem.
ad	S.AB.ad	idem.
bc	S.AB.bc	idem.
bd	S.AB.bd	idem.
cd	S.AB.cd	idem.

14.1.1 Example: square attributes

```
\begin{tkzelements}
z.A         = point  : new ( 0 , 0 )
z.B         = point  : new ( 4 , 0 )
z.C         = point  : new ( 4 , 4)
z.D         = point  : new ( 0 , 4)
S.new       = square : new ( z.A , z.B ,z.C,z.D)
z.I         = S.new.center
z.H         = S.new.proj
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawCircles[orange](I,A I,H)
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D,H,I)
\tkzLabelPoints(A,B,H,I)
\tkzLabelPoints[above](C,D)
\tkzDrawSegments(I,B I,H)
\tkzLabelSegment[sloped](I,B){\pmpn{\tkzUseLua{S.new.exradius}}}
\tkzLabelSegment[sloped](I,H){\pmpn{\tkzUseLua{S.new.inradius}}}
\tkzLabelSegment[sloped](D,C){\pmpn{\tkzUseLua{S.new.side}}}
\end{tikzpicture}
```



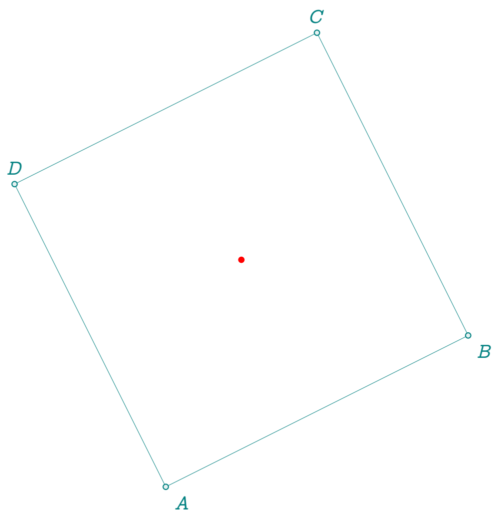
14.2 Square methods

Table 16: Square methods.

Methods	Comments	
`rotation (zi,za)`	`S.IA = square : rotation (z.I,z.A)`	*I* square center *A* first vertex
`side (za,zb)`	`S.AB = square : side (z.A,z.B)`	AB is the first side (direct)

14.2.1 Square with side method

```
\begin{tkzelements}
   scale        = 2
   z.A          = point : new ( 0 , 0 )
   z.B          = point : new ( 2 , 1 )
   S.side       = square : side (z.A,z.B)
   z.B          = S.side.pb
   z.C          = S.side.pc
   z.D          = S.side.pd
   z.I          = S.side.center
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C,D)
   \tkzDrawPoints(A,B,C,D)
   \tkzLabelPoints(A,B)
   \tkzLabelPoints[above](C,D)
   \tkzDrawPoints[red](I)
\end{tikzpicture}
```



15 Class rectangle

15.1 Rectangle attributes

Points are created in the direct direction. A test is performed to check whether the points form a rectangle, otherwise compilation is blocked.

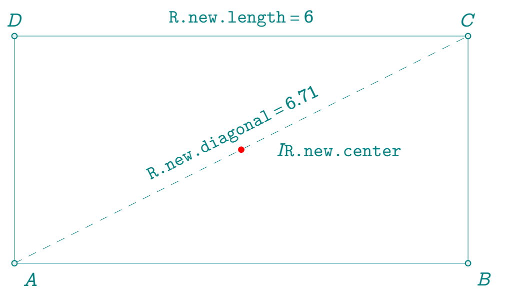
```
Creation  R.ABCD = rectangle : new (z.A,z.B,z.C,z.D)
```

Table 17: rectangle attributes.

Attributes	Application	
pa	z.A = R.ABCD.pa	
pb	z.B = R.ABCD.pb	
pc	z.C = R.ABCD.pc	
pd	z.D = R.ABCD.pd	
type	R.ABCD.type= 'rectangle'	
center	z.I = R.ABCD.center	center of the rectangle
length	R.ABCD.length	the length
width	R.ABCD.width	the width
diagonal	R.ABCD.diagonal	diagonal length
ab	R.ABCD.ab	line passing through two vertices
ac	R.ABCD.ca	idem.
ad	R.ABCD.ad	idem.
bc	R.ABCD.bc	idem.
bd	R.ABCD.bd	idem.
cd	R.ABCD.cd	idem.

15.1.1 Example

```
\begin{tkzelements}
z.A   = point : new ( 0 , 0 )
z.B   = point : new ( 4 , 0 )
z.C   = point  : new ( 4 , 4)
z.D   = point  : new ( 0 , 4)
R.new = rectangle : new (z.A,z.B,z.C,z.D)
z.I   = R.new.center
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D)
\tkzLabelPoints(A,B)
\tkzLabelPoints[above](C,D)
\tkzDrawPoints[red](I)
\end{tikzpicture}
```



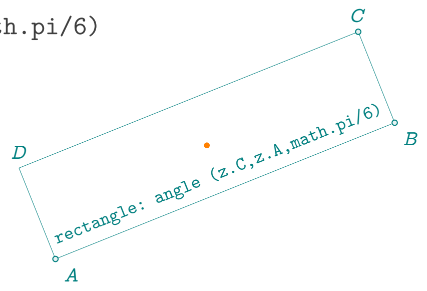
15.2 Rectangle methods

Table 18: Rectangle methods.

Methods	Comments	
`angle (zi,za,angle)`	`R.ang = rectangle : angle (z.I,z.A);z.A`	vertex ; ang angle between 2 vertices
`gold (za,zb)`	`R.gold = rectangle : gold (z.A,z.B)`	length/width = ϕ
`diagonal (za,zc)`	`R.diag = rectangle : diagonal (z.I,z.A)`	I square center A first vertex
`side (za,zb,d)`	`S.IA = rectangle : side (z.I,z.A)`	I square center A first vertex
`get_lengths ()`	`S.IA = rectangle : get_lengths ()`	I square center A first vertex

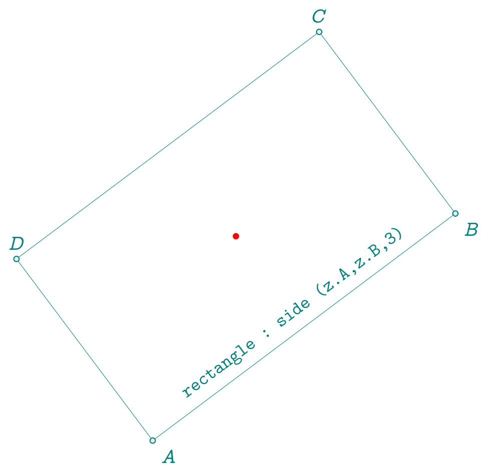
15.2.1 Angle method

```
\begin{tkzelements}
scale    = .5
z.A      = point : new ( 0 , 0 )
z.B      = point : new ( 4 , 0 )
z.I      = point : new ( 4 , 3 )
P.ABCD   = rectangle : angle ( z.I , z.A , math.pi/6)
z.B      = P.ABCD.pb
z.C      = P.ABCD.pc
z.D      = P.ABCD.pd
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C)
\tkzLabelPoints(A,B,C,D)
\tkzDrawPoints[new](I)
\end{tikzpicture}
```



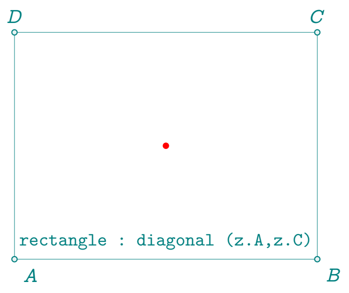
15.2.2 Side method

```
\begin{tkzelements}
z.A    = point : new ( 0 , 0 )
z.B    = point : new ( 4 , 3 )
R.side = rectangle : side (z.A,z.B,3)
z.C    = R.side.pc
z.D    = R.side.pd
z.I    = R.side.center
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D)
\tkzLabelPoints(A,B)
\tkzLabelPoints[above](C,D)
\tkzDrawPoints[red](I)
\end{tikzpicture}
```



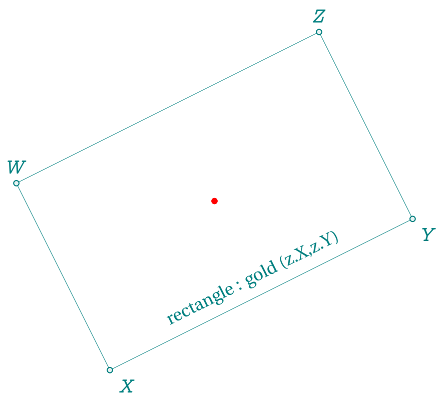
15.2.3 Diagonal method

```
\begin{tkzelements}
z.A         = point : new ( 0 , 0 )
z.C         = point : new ( 4 , 3 )
R.diag      = rectangle : diagonal (z.A,z.C)
z.B         = R.diag.pb
z.D         = R.diag.pd
z.I         = R.diag.center
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D)
\tkzLabelPoints(A,B)
\tkzLabelPoints[above](C,D)
\tkzDrawPoints[red](I)
\tkzLabelSegment[sloped,above](A,B){|rectangle : diagonal (z.A,z.C)|}
\end{tikzpicture}
```



15.2.4 Gold method

```
\begin{tkzelements}
z.X    = point : new ( 0 , 0 )
z.Y    = point : new ( 4 , 2 )
R.gold = rectangle : gold (z.X,z.Y)
z.Z    = R.gold.pc
z.W    = R.gold.pd
z.I    = R.gold.center
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(X,Y,Z,W)
\tkzDrawPoints(X,Y,Z,W)
\tkzLabelPoints(X,Y)
\tkzLabelPoints[above](Z,W)
\tkzDrawPoints[red](I)
\tkzLabelSegment[sloped,above](X,Y){rectangle :  gold (z.X,z.Y)}
\end{tikzpicture}
```



16 Class parallelogram

16.1 Parallelogram attributes

Points are created in the direct direction. A test is performed to check whether the points form a parallelogram, otherwise compilation is blocked.

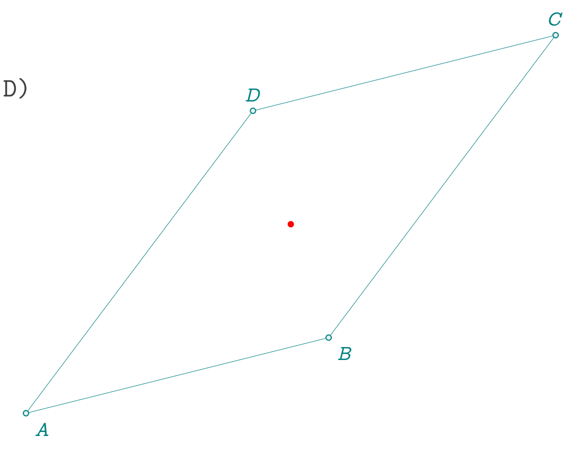
```
Creation P.new = parallelogram : new (z.A,z.B,z.C,z.D)
```

Table 19: Parallelogram attributes.

Attributes	**Application**	
pa	z.A = P.new.pa	
pb	z.B = P.new.pb	
pc	z.C = P.new.pc	
pd	z.D = P.new.pd	
type	P.new.type= 'parallelogram'	
i	z.I = P.new.i	intersection of diagonals
ab	P.new.ab	line passing through two vertices
ac	P.new.ca	idem.
ad	P.new.ad	idem.
bc	P.new.bc	idem.
bd	P.new.bd	idem.
cd	P.new.cd	idem.

16.1.1 Example: attributes

```
\begin{tkzelements}
z.A          = point : new ( 0 , 0 )
z.B          = point : new ( 4 , 1 )
z.C          = point : new ( 7 , 5 )
z.D          = point : new ( 3 , 4 )
P.new        = parallelogram : new (z.A,z.B,z.C,z.D)
z.B          = P.new.pb
z.C          = P.new.pc
z.D          = P.new.pd
z.I          = P.new.center
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D)
\tkzLabelPoints(A,B)
\tkzLabelPoints[above](C,D)
\tkzDrawPoints[red](I)
\end{tikzpicture}
```



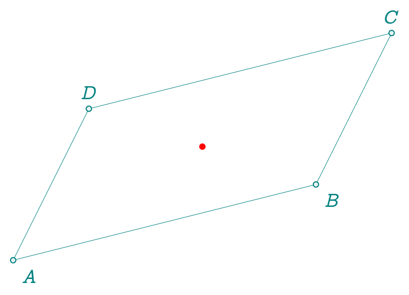
16.2 Parallelogram methods

Table 20: Parallelogram methods.

Methods	Comments
`fourth (za,zb,zc)`	completes a triangle by parallelogram (Refer to next example)

16.2.1 parallelogram with fourth method

```
\begin{tkzelements}
   scale = .75
z.A      = point : new ( 0 , 0 )
z.B      = point : new ( 4 , 1 )
z.C      = point : new ( 5 , 3 )
P.four   = parallelogram : fourth (z.A,z.B,z.C)
z.D      = P.four.pd
z.I      = P.four.center
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C,D)
\tkzDrawPoints(A,B,C,D)
\tkzLabelPoints(A,B)
\tkzLabelPoints[above](C,D)
\tkzDrawPoints[red](I)
\end{tikzpicture}
```



17 Class regular polygon

17.1 regular_polygon attributes

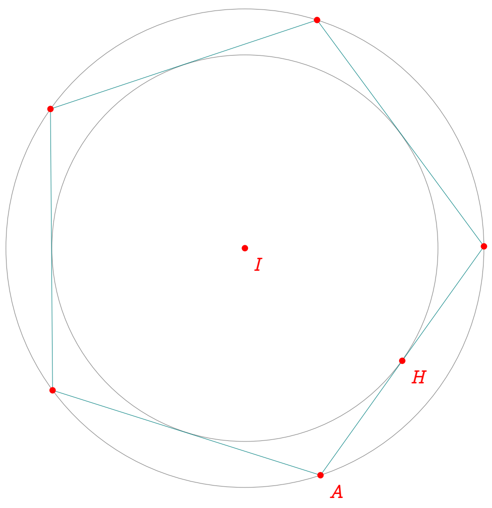
Creation `RP.IA = regular_polygon : new (z.I,z.A,6)`

Table 21: Regular_polygon attributes.

Attributes	Application
`center`	`z.I = RP.IA.center`
`table`	array containing all vertex affixes
`through`	first vertex
`circle`	defines the circle with center I passing through A
`type`	`RP.IA.type= 'regular\_polygon'`
`side`	`s = RP.IA.side` ; s = length of side
`exradius`	`S.AB.exradius` ; radius of the circumscribed circle
`inradius`	`S.AB.inxradius` ; radius of the inscribed circle
`proj`	`RP.IA.proj` ; projection of the center on one side
`angle`	`RP.IA.angle` ; angle formed by the center and 2 consecutive vertices

17.1.1 Pentagon

```
\begin{tkzelements}
z.O     = point:    new (0,0)
z.I     = point:    new (1,3)
z.A     = point:    new (2,0)
RP.five  = regular_polygon : new (z.I,z.A,5)
RP.five : name ("P_")
C.ins  = circle: radius (z.I,RP.five.inradius)
z.H = RP.five.proj
\end{tkzelements}
\begin{tikzpicture}
\def\nb{\tkzUseLua{RP.five.nb}}
\tkzGetNodes
\tkzDrawCircles(I,A I,H)
\tkzDrawPolygon(P_1,P_...,P_\nb)
\tkzDrawPoints[red](P_1,P_...,P_\nb,H,I)
\tkzLabelPoints[red](I,A,H)
\end{tikzpicture}
```



17.2 regular_polygon methods

Table 22: Circle methods.

Methods	Comments
`new(O,A,n)`	`RP.five     = regular_polygon : new (z.I,z.A,5)` ; I center A first vertex 5 sides
Circle	
`incircle ()`	`C.IH = RP.five : incircle ()`
Points	
`name (string)`	Refer to 17.1.1

18 Class vector

In fact, they are more a class of oriented segments than vectors in the strict mathematical sense.
A vector is defined by giving two points (i.e. two affixes). `V.AB = vector : new (z.A,z.B)` creates the vector $\vec(AB)$, i.e. the oriented segment with origin A representing a vector. A few rudimentary operations are defined, such as sum, subtraction and multiplication by a scalar.
The sum is defined as follows:
Let V.AB + V.CD result in a vector V.AE defined as follows
If $\overrightarrow{CD} = \overrightarrow{BE}$ then $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AB} + \overrightarrow{BE} = \vec(AE)$

Creation `V.AB = vector: new (z.A,z.B)`

```
z.A = ...
z.B = ...
z.C = ...
z.D = ...
V.AB = vector : new (z.A,z.B)
V.CD = vector : new (z.C,z.D)
V.AE = V.AB + V.CD  -- possible V.AB : add (V.CD)
z.E  = V.AE.head -- we recover the final point (head)
```

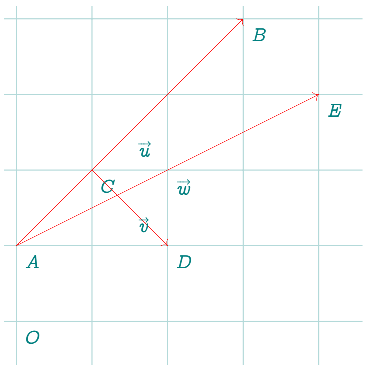
18.1 Attributes of a vector

Table 23: Vector attributes.

Attributes	**Application**	**Example**
`tail`	`V.AB.t = z.A`	Refer to (7.2.2)
`head`	`V.AB.head = z.B`	Refer to (7.2.2)
`type`	`V.AB.type = 'vector'`	
`slope`	`V.AB.slope`	Refer to (18.1.1)
`length`	`V.AB.norm`	Refer to (18.1.1)
`mtx`	`V.AB.mtx`	The result is a column matrix `{{V.AB.t},{V.AB.h}}`

18.1.1 Example vector attributes

```
\begin{tkzelements}
  z.O         = point: new (0,0)
  z.A         = point: new (0,1)
  z.B         = point: new (3,4)
  L.AB        = line : new ( z.A , z.B )
  z.C         = point: new (1,2)
  z.D         = point: new (2,1)
  u           = vector : new (z.A,z.B)
  v           = vector : new (z.C,z.D)
  w =u+v
  z.E = w.head
\end{tkzelements}
\begin{tikzpicture}[gridded]
  \tkzGetNodes
  \tkzLabelPoints(A,B,C,D,O,E)
  \tkzDrawSegments[->,red](A,B C,D A,E)
  \tkzLabelSegment(A,B){$\overrightarrow{u}$}
  \tkzLabelSegment(C,D){$\overrightarrow{v}$}
  \tkzLabelSegment(A,E){$\overrightarrow{w}$}
\end{tikzpicture}
$\overrightarrow{w}$ has slope :
$\tkzDN{\tkzUseLua{math.deg(w.slope)}}^\circ$
```



$\overrightarrow{w}$ has slope : 26.57°

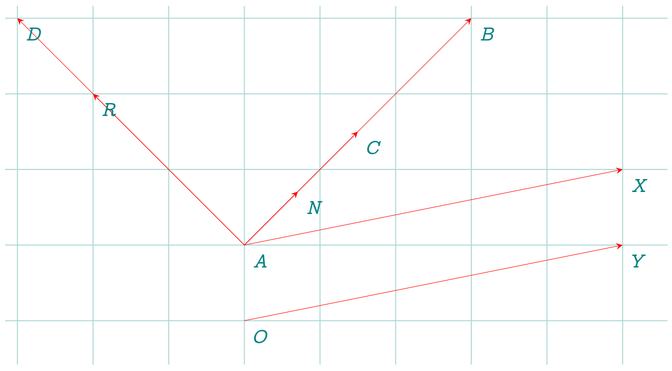
18.2 Methods of the class vector

Table 24: Methods of the class vector.

Metamethods	Application	
`__add (u,v)`	`V.AB + V.CD`	
`__sub (u,v)`	`V.AB - V.CD`	
`__unm (u)`	`V.CD = -V.AB`	
`__mul (k,u)`	`V.CD = k*V.AB`	
Methods	**Application**	
`new(pt, pt)`	`V.AB = vector: new (z.A,z.B)`	
`normalize(V)`	`V.AB : normalize ()`	
`orthogonal(d)`	`V.AB : orthogonal (d)`	
`scale(d)`	`V.CD = V.AB : scale (2)`	$\overrightarrow{CD} = 2\overrightarrow{AB}$
`at (V)`	`V.DB = V.AC : at (z.D)`	$\overrightarrow{DB} = \overrightarrow{AC}$

18.2.1 Example of methods

```
\begin{tkzelements}
  z.O   = point: new (0,0)
  z.A   = point: new (0,1)
  z.B   = point: new (3,4)
  V.AB  = vector: new (z.A,z.B)
  V.AC  = V.AB : scale (.5)
  z.C   = V.AC.head
  V.AD  = V.AB : orthogonal ()
  z.D   = V.AD.head
  V.AN  = V.AB : normalize ()
  z.N   = V.AN.head
  V.AR  = V.AB : orthogonal(2*math.sqrt(2))
  z.R   = V.AR.head
  V.AX  = 2*V.AC - V.AR
  z.X   = V.AX.head
  V.OY  = V.AX : at (z.O)
  z.Y   = V.OY.head
\end{tkzelements}
\begin{tikzpicture}[gridded]
  \tkzGetNodes
  \tkzDrawSegments[>=stealth,->,red](A,B A,C A,D A,N A,R A,X O,Y)
  \tkzLabelPoints(A,B,C,D,O,N,R,X,Y)
\end{tikzpicture}
```



19 Class matrix

The `matrix` class is currently experimental, and its attribute and method names have not yet been finalized, indicating that this class is still evolving. Certain connections have been made with other classes, such as the `point` class. Additionally, a new attribute, `mtx`, has been included, associating a column matrix with the point, where the elements correspond to the point's coordinates in the original base. Similarly, an attribute has been added to the `vector` class, where `mtx` represents a column matrix consisting of the two affixes that compose the vector.
This `matrix` class has been created to avoid the need for an external library, and has been adapted to plane transformations. It allows you to use complex numbers.
☞ To display matrices, you'll need to load the `amsmath` package.
☞ While some methods are valid for any matrix size, the majority are reserved for square matrices of order 2 and 3.

19.1 Matrix creation

– The first method is: (Refer to 19.5.1)

```
M = matrix: new ({{a,b},{c,d}})
or M = matrix: new {{a,b},{c,d}}
a, b, c, et d being real or complex numbers.
```

$$M = \begin{bmatrix} 3 & 2.25 \\ 4 & 3.90 \end{bmatrix}$$

– It is also possible to obtain a square matrix with: (Refer to 19.5.7)

```
M = matrix : square (2,a,b,c,d)
```

– In the case of a column vector: (Refer to 19.5.2)

```
V = matrix : vector (1,2,3)
```

$$V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

– Homogeneous transformation matrix (Refer to 19.5.4)

The objective is to generate a matrix with homogeneous coordinates capable of transforming a coordinate system through rotation, translation, and scaling. To achieve this, it is necessary to define both the rotation angle, the coordinates of the new origin ans the scaling factors.

```
H = matrix : htm (math.pi/3,1,2,2,1)
```

$$H = \begin{bmatrix} 1 & -0.87 & 1 \\ 0.87 & 0.50 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

19.2 Display a matrix: method print

This method (Refer to 19.5.8) is necessary to control the results, so here are a few explanations on how to use it. It can be used on real or complex matrices, square or not. A few options allow you to format the results. You need to load the `amsmath` package to use the "print" method. Without this package, it is possible to display the contents of the matrix without formatting with `print_array (M)`

```
\begin{tkzelements}
M = matrix : new {{1,-1},{2,0}}
M : print ()
\end{tkzelements}
```

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

19.3 Attibutes of a matrix

Table 25: Matrix attributes.

Attributes	Application	
set	`M.set = {{a,b},{c,d}}`	table of tables
rows	`M.rows`	number of rows
cols	`M.cols`	number of columns
type	`M.type = "matrix"`	the type of object
det	`M.det`	determinant of a square matrix or `nil`

19.3.1 Attribute set

A simple array such as `{{1,2},{2,-1}}` is often considered a "matrix". In "tkz-elements", we'll consider `M` defined by `matrix : new ({{1,1},{0,2}})` as a matrix and `M.set` as an array (`M.set = {{1,1},{0,2}}`).
You can access a particular element of the matrix, for example: `M.set[2][1]` gives 0.
`\tkzUseLua{M.set[2][1]}` is the expression that displays 2.
The number of rows is accessed with `M.rows` and the number of columns with `M.cols`, here's an example:

```
\begin{tkzelements}
M = matrix : new ({{1,2,3},{4,5,6}})
M : print ()
tex.print("Rows:  "..M.rows)
tex.print("Cols:  "..M.cols)
\end{tkzelements}
```

$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$Rows: 2 Cols: 3

19.3.2 Determinant with real numbers

The matrix must be square. This library was created for matrices of dimension 2 or 3, but it is possible to work with larger sizes. `det` is an attribute of the "matrix" object, but the determinant can also be obtained with the function `determinant(M)`.

```
  \begin{tkzelements}
  M = matrix : square (3,1,1,0,2,-1,-2,1,-1,2)
  M : print ()
  tex.print ('\\\\')
  tex.print ("Its determinant is:  " .. M.det)
  \end{tkzelements}
```

$$\begin{bmatrix}1 & 1 & 0\\2 & -1 & -2\\1 & -1 & 2\end{bmatrix}$$
Its determinant is: -10.0

19.3.3 Determinant with complex numbers

```
\begin{tkzelements}
 a = point :new (1,-2)
 b = point :new (0,1)
 c = point :new (1,1)
 d = point :new (1,-1)
 A = matrix : new ({{a, b}, {c,d}})
 tex.print(tostring(A.det))
\end{tkzelements}
```

-4.00i

19.4 Metamethods for the matrices

Conditions on matrices must be valid for certain operations to be possible.

Table 26: Matrix metamethods.

Metamethods	Application	
`__add(M1,M2)`	`M1 + M2`	
`__sub(M1,M2)`	`M1 - M2`	
`__unm(M`	`- M`	
`__mul(M1,M2)`	`M1 * M2`	
`__pow(M,n)`	`M ^ n`	n integer > or < 0 or `'T'`
`__tostroing(M,n)`	tex.print(tostring(M))	displays the matrix
`__eq(M1,M2)`	true or false	

19.4.1 Addition and subtraction of matrices

To simplify the entries, I've used a few functions to simplify the displays.

```
\begin{tkzelements}
  A = matrix : new ({{1,2},{2,-1}})
  B = matrix : new ({{-1,0},{1,3}})
  S = A + B
  D = A - B
  dsp(A,'A')
  nl() nl()
  dsp(B,'B')
  nl() nl()
  dsp(S,'S') sym(" = ") dsp(A) sym(' + ') dsp(B)
  nl() nl()
  dsp(D,'D') sym(" = ") dsp(A) sym(' - ') dsp(B)
\end{tkzelements}
```

$$\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\mathrm{B} = \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}$$

$$\mathrm{S} = \begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}$$

$$\mathrm{D} = \begin{bmatrix} 2 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}$$

19.4.2 Multiplication and power of matrices

To simplify the entries, I've used a few functions. You can find their definitions in the sources section of this documentation.

```
  \begin{tkzelements}
    A = matrix : new ({{1,2},{2,-1}})
    B = matrix : new ({{-1,0},{1,3}})
    P = A * B
    I = A^-1
    C = A^3
    K = 2 * A
    T = A^'T'
  \end{tkzelements}
```

$$\mathrm{P} = \begin{bmatrix} 1 & 6 \\ -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} * \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0.20 & 0.40 \\ 0.40 & -0.20 \end{bmatrix}$$

$$\mathrm{K} = \begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

19.4.3 Metamethod eq

Test whether two matrices are equal or identical.

19.5 Methods of the class matrix

Table 27: Matrix methods.

Functions	Comments	
new(...)	M = matrix : new ({{1,2},{2,-1}})	
square()	M = matrix : square (2,1,2,2,-1)	
vector()	M = matrix : vector (2,1)	
htm()	M = matrix : htm (2,1,2,2,-1)	
Methods	**Comments**	
print(s,n)	M : print ()	s='matrix' or ... default 'bmatrix'
htm_apply(...)	M : htm_apply (...)	
get()	M : get (i,j)	i = raws , j = cols Refer to 19.5.10
inverse()	M : inverse ()	
adjugate()	M : adjugate ()	
transpose()	M : transpose ()	
is_diagonal()	true or false	result :boolean
is_orthogonal()	true or false	
homogenization()	M : homogenization ()	

19.5.1 Function new

This is the main method for creating a matrix. Here's an example of a 2x3 matrix with complex coefficients:

```
\begin{tkzelements}
a = point : new (1,0)
b = point : new (1,1)
c = point : new (-1,1)
d = point : new (0,1)
e = point : new (1,-1)
f = point : new (0,-1)
M = matrix : new ({{a,b,c},{d,e,f}})
M : print ()
\end{tkzelements}
```

$$\begin{bmatrix} 1 & 1+i & -1+i \\ i & 1-i & i \end{bmatrix}$$

19.5.2 Function vector

The special case of a column matrix, frequently used to represent a vector, can be treated as follows:

```
\begin{tkzelements}
M = matrix : vector (1,2,3)
M : print ()
\end{tkzelements}
```

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

19.5.3 Method homogenization

homogenization of vector: the aim is to be able to use a homogeneous transformation matrix
Let's take a point A such that z.A = point : new (2,-1). In order to apply a htm matrix, we need to perform a few operations on this point. The first is to determine the vector (matrix) associated with the point. This is straightforward, since there's a point attribute called mtx which gives this vector:

```
z.A = point : new (2,-1)
V = z.A.mtx : homogenization ()
```

which gives:

```
\begin{tkzelements}
  pi   = math.pi
  M    = matrix : htm (pi/4 , 3 , 1)
  z.A = point : new (2,-1)
  V    = z.A.mtx : homogenization ()
  z.A.mtx : print ()
  tex.print ('then after homogenization: ')
  V : print ()
\end{tkzelements}
```

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ then after homogenization: $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

19.5.4 Function htm: homogeneous transformation matrix

There are several ways of using this transformation. First, we need to create a matrix that can associate a rotation with a translation.
The main method is to create the matrix:

```
pi = math.pi
M = matrix : htm (pi/4 , 3 , 1)
```

A 3x3 matrix is created which combines a $\pi/4$ rotation and a $\vec{t} = (3,1)$ translation.

$$\begin{bmatrix} 0.71 & -0.71 & 3 \\ 0.71 & 0.71 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can apply the matrix M. Let A be the point defined here: 19.5.3. By homogenization, we obtain the column matrix V.

```
W = A * V
```

$$\begin{bmatrix} 0.71 & -0.71 & 3 \\ 0.71 & 0.71 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.12 \\ 1.71 \\ 1 \end{bmatrix}$$

All that remains is to extract the coordinates of the new point.

19.5.5 Method get_htm_point

In the previous section, we obtained the W matrix. Now we need to obtain the point it defines.
The method `get_htm_point` extracts a point from a vector obtained after applying a `htm` matrix.

```
\begin{tkzelements}
  W : print ()
  z.P = get_htm_point(W)
  tex.print("The affix of $P$ is: ")
  tex.print(display(z.P))
\end{tkzelements}
```

$\begin{bmatrix} 5.12 \\ 1.71 \\ 1 \end{bmatrix}$ The affix of P is: 5.12+1.71i

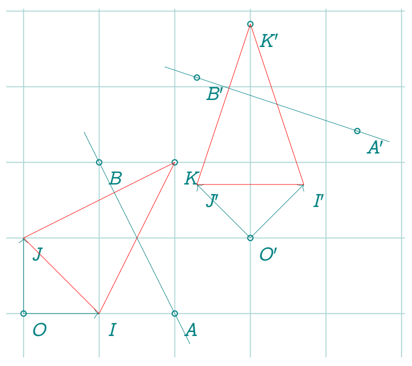
19.5.6 Method htm_apply

The above operations can be simplified by using the `htm_apply` method directly at point A.

```
z.Ap = M: htm_apply (z.A)
```

Then the method `htm_apply` transforms a point, a list of points or an object.

```
\begin{tkzelements}
  pi        = math.pi
  M         = matrix : htm (pi/4 , 3 , 1 )
  z.O       = point : new (0,0)
  V.ori     = z.O.mtx : homogenization ()
  z.I       =  point : new (1,0)
  z.J       =  point : new (0,1)
  z.A       = point: new (2,0)
  z.B       = point: new (1,2)
  L.AB      = line : new (z.A,z.B)
  z.Op,z.Ip,z.Jp = M : htm_apply (z.O,z.I,z.J)
  L.ApBp    = M : htm_apply (L.AB)
  z.Ap      = L.ApBp.pa
  z.Bp      = L.ApBp.pb
  z.K       = point : new (2,2)
  T        = triangle : new ( z.I , z.J , z.K )
  Tp        =  M : htm_apply (T)
  z.Kp     = Tp.pc
\end{tkzelements}
```



New cartesian coordinates system:

```
\begin{tkzelements}
  pi = math.pi
  tp = tex.print
  nl = '\\\\'
  a = point(1,0)
  b = point(0,1)
  R = matrix : htm (pi/5,2,1)
  R : print () tp(nl)
  v = matrix : vector (1,2)
  v : print ()
  v.h = v : homogenization ()
  v.h : print () tp(nl)
  V =   R * v.h
  V : print ()
  z.N = get_htm_point(V)
  tex.print(display(z.N))
\end{tkzelements}
```

$$\begin{bmatrix} 0.81 & -0.59 & 2 \\ 0.59 & 0.81 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1.63 \\ 3.21 \\ 1 \end{bmatrix} 1.63+3.21i$$

19.5.7 Function square

We have already seen this method in the presentation of matrices. We first need to give the order of the matrix, then the coefficients, row by row.

```
\begin{tkzelements}
M = matrix : square (2,2,3,-5,4)
M : print ()
\end{tkzelements}
```

$$\begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix}$$

19.5.8 Method print

With the amsmath package loaded, this method can be used. By default, the bmatrix environment is selected, although you can choose from matrix, pmatrix, Bmatrix, "vmatrix", "Vmatrix". Another option lets you set the number of digits after the decimal point. The "tkz_dc" global variable is used to set the number of decimal places. Here's an example:

```
\begin{tkzelements}
    M = matrix : new ({{math.sqrt(2),math.sqrt(3)},{math.sqrt(4),math.sqrt(5)}})
    M : print ('pmatrix')
\end{tkzelements}
```

$$\begin{pmatrix} 1.414 & 1.732 \\ 2 & 2.236 \end{pmatrix}$$

You can also display the matrix as a simple array using the `print_array (M)` function. refer to the next example. In the case of a square matrix, it is possible to transmit a list of values whose first element is the order of the matrix.

```
    \begin{tkzelements}
    M = matrix : square (2,1,0,0,2)
    M : print ()
    \end{tkzelements}
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

19.5.9 Display a table or array: function print_array

We'll need to display results, so let's look at the different ways of displaying them, and distinguish the differences between arrays and matrices.
Below, A is an array. It can be displayed as a simple array or as a matrix, but we can't use the attributes and `A : print ()` is not possible because A is not an object of the class `matrix`. If you want to display an array like a matrix you can use the function `print_matrix` (refer to the next example).

```
\begin{tkzelements}
  A = {{1,2},{1,-1}}
  tex.print ('A = ') print_array (A)
  tex.print (' or ')
  print_matrix (A)
  M = matrix : new ({{1,1},{0,2}})
  tex.print ('\\\\')
  tex.print ('M = ')  M : print ()
\end{tkzelements}
```

A = {{ 1 , 2 },{ 1 , -1 }}or $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

M = $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

19.5.10 Get an element of a matrix: method get

```
  \begin{tkzelements}
  M = matrix : new {{1,2},{2,-1}}
  S = M: get(1,1) + M: get(2,2)
  tex.print(S)
  \end{tkzelements}
```

0

19.5.11 Inverse matrix: : method inverse

```
\begin{tkzelements}
 A = matrix : new ({{1,2},{2,-1}})
 tex.print("Inverse of $A = $")
 B =  A : inverse ()
 B : print ()
\end{tkzelements}
```

Inverse of $A = \begin{bmatrix} 0.43 & 0.29 \\ 0.29 & -0.14 \end{bmatrix}$

19.5.12 Inverse matrix with power syntax

```
\begin{tkzelements}
  M = matrix : new ({{1,0,1},{1,2, 1},{0,-1,2}})
  tex.print("$M = $")  print_matrix (M)
  tex.print('\\\\')
  tex.print("Inverse of $M = M^{-1} = $")
  print_matrix (M^-1)
\end{tkzelements}
```

$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

Inverse of $M = M^{-1} =$ $\begin{bmatrix} 1.25 & -0.25 & -0.50 \\ -0.50 & 0.50 & 0 \\ -0.25 & 0.25 & 0.50 \end{bmatrix}$

19.5.13 Transpose matrix: method transpose

A transposed matrix can be accessed with `A: transpose ()` or with `A^{'T'}`.

```
    \begin{tkzelements}
      A = matrix : new ({{1,2},{2,-1}})
      AT = A : transpose ()
      tex.print("$A^{'T'} = $")
       AT : print ()
    \end{tkzelements}
```

$A^{'T'} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Remark: `(A ^'T')^'T' = A`

19.5.14 Method method adjugate

```
\begin{tkzelements}
  N =  matrix : new {{1, 0, 3},{2, 1, 0},{-1, 2, 0}}
  tex.print('N = ') print_matrix(N)
  tex.print('\\\\')
  N.a = N : adjugate ()
  N.i = N * N.a
  tex.print('adj(N) = ') N.a : print ()
  tex.print('\\\\')
  tex.print('N $\\times$ adj(N) = ') print_matrix(N.i)
  tex.print('\\\\')
  tex.print('det(N) = ') tex.print(N.det)
\end{tkzelements}
```

N = $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ adj(N) = $\begin{bmatrix} 0 & 6 & -3 \\ 0 & 3 & 6 \\ 5 & -2 & 1 \end{bmatrix}$

N × adj(N) = $\begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

det(N) = 15.0

19.5.15 Method method identity

Creating the identity matrix order 3

```
  \begin{tkzelements}
  Id_3 = matrix : identity (3)
  Id_3 : print ()
  \end{tkzelements}
```

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19.5.16 Diagonalization: method diagonalize

For the moment, this method only concerns matrices of order 2.

```
\begin{tkzelements}
A = matrix : new  {{5,-3}, {6,-4}}
 tex.print('A = ') A : print ()
 D,P =  A : diagonalize ()
 tex.print('D = ')  D : print ()
 tex.print('P = ') P : print ()
 R = P^(-1)*A*P
 tex.print('\\\\')
 tex.print('Test: $D = P^{-1}AP = $ ')
  R : print ()
   tex.print('\\\\')
 tex.print('Verification: $P^{-1}P = $ ')
 T = P^(-1)*P
 T  : print ()
\end{tkzelements}
```

A = $\begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}$D = $\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$P = $\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$

Test: $D = P^{-1}AP = \begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}$

Verification: $P^{-1}P = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

19.5.17 Method is_orthogonal

The method returns `true` if the matrix is orthogonal and `false` otherwise.

```
\begin{tkzelements}
local  cos = math.cos
local  sin = math.sin
local  pi  = math.pi
A = matrix : new ({{cos(pi/6),-sin(pi/6)}, {sin(pi/6),cos(pi/6)}})
A : print ()
bool = A : is_orthogonal ()
tex.print('\\\\')
if bool
then
  tex.print("The matrix is orthogonal")
else
tex.print("The matrix is not orthogonal")
end
tex.print('\\\\')
tex.print('Test: $A^T = A^{-1} ?$')
print_matrix(transposeMatrix (A))
tex.print('=')
inv_matrix (A) : print ()
\end{tkzelements}
```

$\begin{bmatrix}0.87 & -0.50\\0.50 & 0.87\end{bmatrix}$

The matrix is orthogonal

Test: $A^T = A^{-1}?\begin{bmatrix}0.87 & 0.50\\-0.50 & 0.87\end{bmatrix} = \begin{bmatrix}0.87 & 0.50\\-0.50 & 0.87\end{bmatrix}$

19.5.18 Method is_diagonal

The method returns `true` if the matrix is diagonal and `false` otherwise.

20 Math constants and functions

Table 28: Math constants and functions.

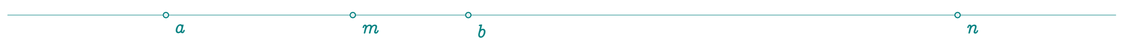
contants or functions	Comments
`tkzphi`	constant $\varphi = (1 + math.sqrt(5))/2$
`tkzinvphi`	constant $1/\varphi = 1/tkzphi$
`tkzsqrtphi`	constant $\sqrt{\varphi} = math.sqrt(tkzphi)$
`length (a,b)`	point.abs(a-b) Refer to (8.2.2)
`islinear (z1,z2,z3)`	Are the points aligned? (z2-z1) || (z3-z1) ?
`isortho (z1,z2,z3)`	(z2-z1) ⊥ (z3-z1) ? boolean
`get_angle (z1,z2,z3)`	the vertex is z1 Refer to (20.8)
`bisector (z1,z2,z3)`	L.Aa = bisector (z.A,z.B,z.C) from A (20.8)
`bisector_ext (z1,z2,z3)`	L.Aa = bisector_ext (z.A,z.B,z.C) from A
`altitude (z1,z2,z3)`	altitude from z1
`set_lua_to_tex (list)`	set_lua_to_tex('a','n') defines `\a` and `\n`
`tkzUseLua (variable)`	`\textbackslash\tkzUseLua{a}` prints the value of a
`value (v)`	apply `scale * value`
`real (v)`	apply `value /scale`
`angle_normalize (an)`	to get a value between 0 and 2π
`barycenter ({z1,n1},{z2,n2}, ...)`	barycenter of list of points
`solve_quadratic (a,b,c)`	gives the solution of $ax^2 + bx + c = 0$ a,b,c real or complex (Refer to 20.12.1)

20.1 Length of a segment

`length(z.A,z.B)` is a shortcut for `point.abs(z.A-z.B)`. This avoids the need to use complexes.

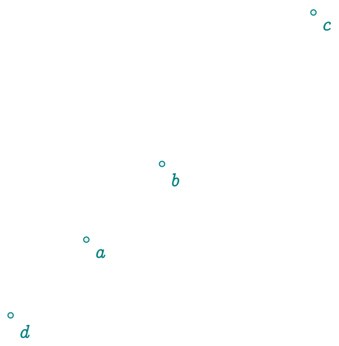
20.2 Harmonic division with tkzphi

```
\begin{tkzelements}
   scale =.5
   z.a = point: new(0,0)
   z.b = point: new(8,0)
   L.ab = line: new (z.a,z.b)
   z.m,z.n = L.ab: harmonic_both (tkzphi)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLine[add= .2 and .2](a,n)
   \tkzDrawPoints(a,b,n,m)
   \tkzLabelPoints(a,b,n,m)
\end{tikzpicture}
```



20.3 Function islinear

```
\begin{tkzelements}
   z.a = point: new (1, 1)
   z.b = point: new (2, 2)
   z.c = point: new (4, 4)
   if islinear (z.a,z.b,z.c) then
       z.d = point: new (0, 0)
    else
        z.d = point: new (-1, -1)
   end
\end{tkzelements}
\begin{tikzpicture}
    \tkzGetNodes
    \tkzDrawPoints(a,...,d)
    \tkzLabelPoints(a,...,d)
\end{tikzpicture}
```



20.4 Function value

value to apply scaling if necessary
If `scale = 1.2` with a = value(5) the actual value of a will be $5 \times 1.2 = 6$.

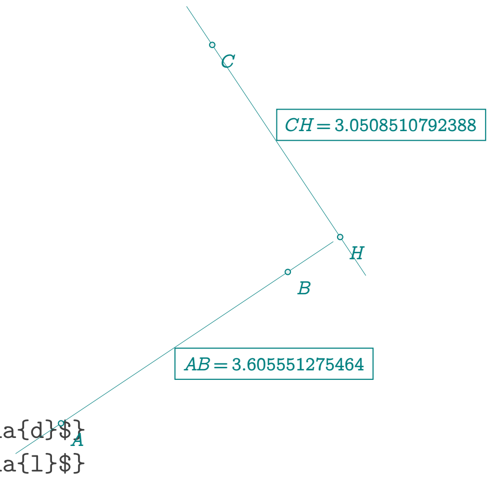
20.5 Function real

If `scale = 1.2` with a = 6 then real(a) = $6/1.2 = 5$.

20.6 Transfer from lua to TEX

It's possible to transfer variable from Lua to TEX with `\tkzUseLua`.

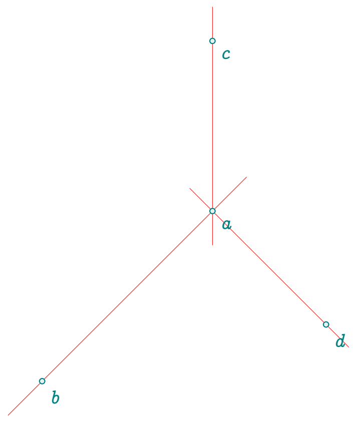
```
\begin{tkzelements}
   z.A              = point : new (0 , 0)
   z.B              = point : new (3 , 2)
   z.C              = point : new (2 , 5)
   L.AB             = line : new (z.A,z.B)
   d                = L.AB : distance (z.C)
   l                = L.AB.length
   z.H              = L.AB : projection (z.C)
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawLines(A,B C,H)
\tkzDrawPoints(A,B,C,H)
\tkzLabelPoints(A,B,C,H)
\tkzLabelSegment[above right,draw](C,H){$CH = \tkzUseLua{d}$}
\tkzLabelSegment[below right,draw](A,B){$AB = \tkzUseLua{l}$}
\end{tikzpicture}
```



20.7 Normalized angles : Slope of lines (ab), (ac) and (ad)

```
\begin{tkzelements}
   z.a      = point: new(0, 0)
   z.b      = point: new(-3, -3)
   z.c      = point: new(0, 3)
   z.d      = point: new(2, -2)
   angle    = point.arg (z.b-z.a)
   tex.print('slope of (ab) : '..tostring(angle)..'\\\\')
   tex.print('slope normalized of (ab) : '..tostring(angle\_normalize(angle))..'\\\\')
   angle    = point.arg (z.c-z.a)
   tex.print('slope of (ac) : '..tostring(angle)..'\\\\')
   tex.print('slope normalized of (ac) : '..tostring(angle\_normalize(angle))..'\\\\')
   angle    = point.arg (z.d-z.a)
   tex.print('slope of (ad) : '..tostring(angle)..'\\\\')
   tex.print('slope normalized of (acd) : '..tostring(angle\_normalize(angle))..'\\\\')
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines[red](a,b a,c a,d)
   \tkzDrawPoints(a,b,c,d)
   \tkzLabelPoints(a,b,c,d)
\end{tikzpicture}
```

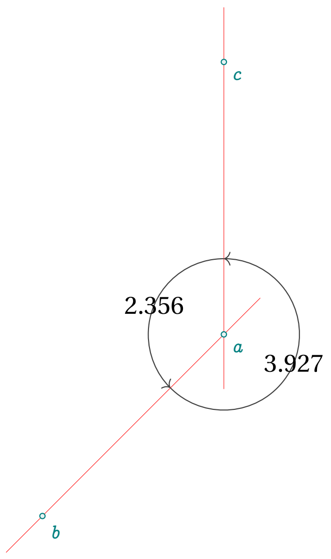
slope of (ab) : -2.3561944901923
slope normalized of (ab) : 3.9269908169872
slope of (ac) : 1.5707963267949
slope normalized of (ac) : 1.5707963267949
slope of (ad) : -0.78539816339745
slope normalized of (ad) : 5.4977871437821



20.8 Get angle

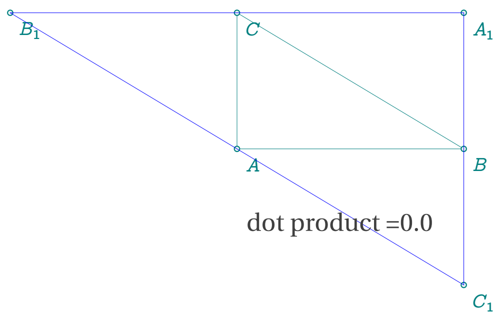
The function `get_angle (a,b,c)` gives the angle normalized of $(\overrightarrow{ab},\overrightarrow{ac})$.

```
\begin{tkzelements}
   z.a   = point: new(0, 0)
   z.b   = point: new(-2, -2)
   z.c   = point: new(0, 3)
   angcb = tkzround ( get_angle (z.a,z.c,z.b),3)
   angbc = tkzround ( get_angle (z.a,z.b,z.c),3)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines[red](a,b a,c)
   \tkzDrawPoints(a,b,c)
   \tkzLabelPoints(a,b,c)
   \tkzMarkAngle[->](c,a,b)
   \tkzLabelAngle(c,a,b){\tkzUseLua{angcb}}
   \tkzMarkAngle[->](b,a,c)
   \tkzLabelAngle(b,a,c){\tkzUseLua{angbc}}
\end{tikzpicture}
```



20.9 Dot or scalar product

```
\begin{tkzelements}
   z.A    = point: new(0,0)
   z.B    = point: new(5,0)
   z.C    = point: new(0,3)
   T.ABC  = triangle: new (z.A,z.B,z.C)
   z.A_1,
   z.B_1,
   z.C_1  = get_points (T.ABC: anti ())
   x  = dot_product (z.A,z.B,z.C)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawPoints(A,B,C,A_1,B_1,C_1)
   \tkzLabelPoints(A,B,C,A_1,B_1,C_1)
   \tkzDrawPolygon[blue](A_1,B_1,C_1)
   \tkzText[right](0,-1){dot product =\tkzUseLua{x}}
\end{tikzpicture}
```



The scalar product of the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ is equal to 0.0, so these vectors are orthogonal.

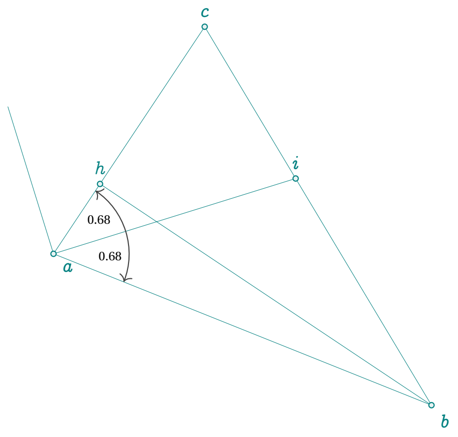
20.10 Alignment or orthogonality

With the functions `islinear` and `isortho`. `islinear(z.a,z.b,z.c)` gives `true` idf the points a, b and c are aligned.
`isortho(z.a,z.b,z.c)` gives `true` if the line (ab) is orthogonal to the line (ac).

20.11 Bisector and altitude

These functions are useful if you don't need to create a useful triangle object for the rest of your code.

```
\begin{tkzelements}
   z.a   = point: new (0, 0)
   z.b   = point: new (5, -2)
   z.c   = point: new (2, 3)
   z.i   = bisector (z.a,z.c,z.b).pb
   z.h   = altitude (z.b,z.a,z.c).pb
   angic = tkzround ( get_angle (z.a,z.i,z.c),2)
   angci = tkzround ( get_angle (z.a,z.b,z.i),2)
   z.e = bisector_ext (z.a,z.b,z.c).pb
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(a,b,c)
   \tkzDrawSegments(a,i b,h a,e)
   \tkzDrawPoints(a,b,c,i,h)
   \tkzLabelPoints(a,b)
   \tkzLabelPoints[above](c,i,h)
   \tkzMarkAngle[->](i,a,c)
   \tkzLabelAngle[font=\tiny,pos=.75](i,a,c){\tkzUseLua{angci}}
   \tkzMarkAngle[<-](b,a,i)
   \tkzLabelAngle[font=\tiny,pos=.75](b,a,i){\tkzUseLua{angic}}
\end{tikzpicture}
```



20.12 Other functions

Not documented because still in beta version: parabola, Cramer22, Cramer33.

20.12.1 Function solve_quadratic

This function solves the equation $ax^2+bx+c=0$ with real or complex numbers.

```
\begin{tkzelements}
    tex.sprint('Solve : $x^2+1=0$ The solution set is  ')
    r1,r2 = solve_quadratic(1,0,1)
    tex.print('\\{'..tostring(r1)..' , '..tostring(r2)..'\\}')
    tex.print('\\\\')
    tex.sprint('Solve : $x^2+2x-3=0$ The solution set is  ')
    r1,r2 = solve_quadratic(1,2,-3)
    tex.print('\\{'..tostring(r1)..' , '..tostring(r2)..'\\}')
    tex.print('\\\\')
    a = point (0,1)
    b = point (1,1)
    c = point (-1,1)
    tex.sprint('Solve : $ix^2+(1+i)x+(-1+i)=0$ The solution set is  ')
    r1,r2 = solve_quadratic(a,b,c)
    tex.print('\\{'..tostring(r1)..' , '..tostring(r2)..'\\}')
\end{tkzelements}
```

Solve : $x^2+1=0$ The solution set is {i , -i}
Solve : $x^2+2x-3=0$ The solution set is {1.0 , -3.0}
Solve : $ix^2+(1+i)x+(-1+i)=0$ The solution set is {0.13-0.68i , -1.13+1.68i}

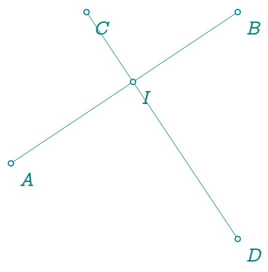
21 Intersections

It's an essential tool. For the moment, the classes concerned are lines, circles and ellipses, with the following combinations: line-line; line-circle; circle-circle and line-ellipse. The argument is a pair of objects, in any order. Results consist of one or two values, either points, boolean **false** or underscore _.

21.1 Line-line

The result is of the form: `point` or `false`.

```
\begin{tkzelements}
   z.A   = point : new (1,-1)
   z.B   = point : new (4,1)
   z.C   = point : new (2,1)
   z.D   = point : new (4,-2)
    z.I  = point : new (0,0)
   L.AB  = line : new (z.A,z.B)
   L.CD  = line : new (z.C,z.D)
   x     = intersection (L.AB,L.CD)
   if x  == false then
   tex.print('error')
   else
   z.I   = x
   end
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawSegments(A,B C,D)
   \tkzDrawPoints(A,B,C,D,I)
   \tkzLabelPoints(A,B,C,D,I)
\end{tikzpicture}
```

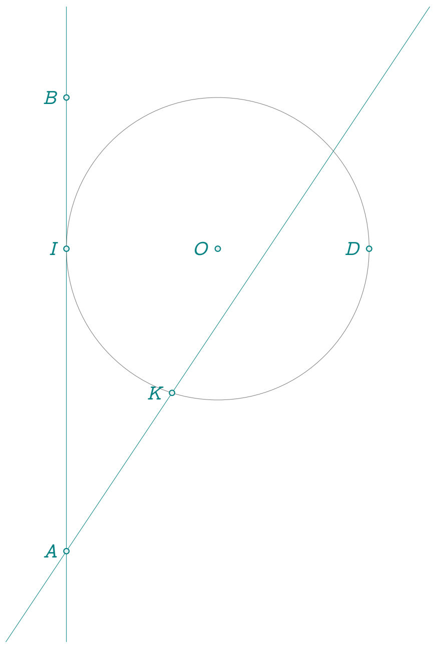


Other examples: 10.2.1, 10.2.2, 23.3

21.2 Line-circle

The result is of the form : `point,point` or `false,false`. If the line is tangent to the circle, then the two points are identical. You can ignore one of the points by using the underscore: `_,point` or `point,_`. When the intersection yields two solutions, the order of the points is determined by the argument of `(z.p - z.c)` with `c` center of the circle and `p` point of intersection. The first solution corresponds to the smallest argument (arguments are between 0 and 2π).

```
\begin{tkzelements}
   z.A    = point : new (1,-1)
   z.B    = point : new (1,2)
   L.AB   = line : new (z.A,z.B)
   z.O    = point : new (2,1)
   z.D    = point : new (3,1)
   z.E    = point : new (3,2)
   L.AE   = line : new (z.A,z.E)
   C.OD   = circle : new (z.O,z.D)
   z.I,_  = intersection (L.AB,C.OD)
   _,z.K  = intersection (C.OD,L.AE)
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
   \tkzDrawLines(A,B A,E)
   \tkzDrawCircle(O,D)
   \tkzDrawPoints(A,B,O,D,I,K)
   \tkzLabelPoints[left](A,B,O,D,I,K)
\end{tikzpicture}
```

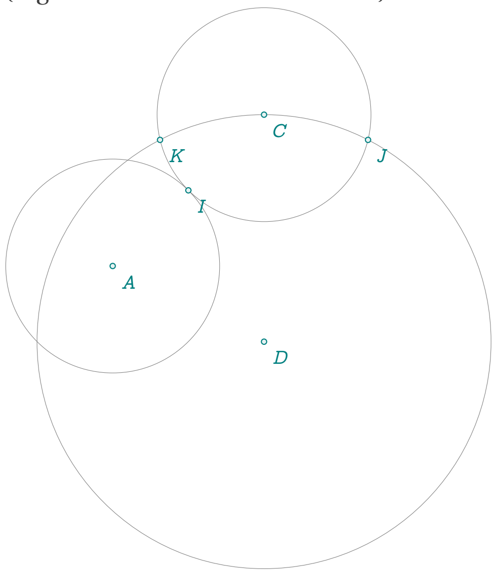


Other examples: 10.2.1

21.3 Circle-circle

The result is of the form : `point,point` or `false,false`. If the circles are tangent, then the two points are identical. You can ignore one of the points by using the underscore: `_ , point` or `point , _`. As for the intersection of a line and a circle, consider the argument of `z.p-z.c` with `c` center of the first circle and `p` point of intersection. The first solution corresponds to the smallest argument (arguments are between 0 and 2π).

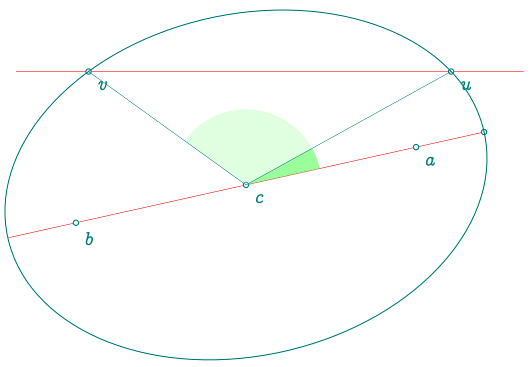
```
\begin{tkzelements}
   z.A      = point : new (1,1)
   z.B      = point : new (2,2)
   z.C      = point : new (3,3)
   z.D      = point : new (3,0)
   C.AB     = circle : new (z.A,z.B)
   C.CB     = circle : new (z.C,z.B)
   z.I,_    = intersection (C.AB,C.CB)
   C.DC     = circle : new (z.D,z.C)
   z.J,z.K  = intersection (C.DC,C.CB)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,B C,B D,C)
   \tkzDrawPoints(A,I,C,D,J,K)
   \tkzLabelPoints(A,I,C,D,J,K)
\end{tikzpicture}
```



Other examples: 10.2.1, 3.3

21.4 Line-ellipse

The following example is complex, but it shows the possibilities of Lua. The designation of intersection points is a little more complicated than the previous one, as the argument characterizing the major axis must be taken into account. The principle is the same, but this argument must be subtracted. In concrete terms, you need to consider the slopes of the lines formed by the center of the ellipse and the points of intersection, and the slope of the major axis.



```
\begin{tkzelements}
   scale     = .5
   z.a       = point: new (5 , 2)
   z.b       = point: new (-4 , 0)
   z.m       = point: new (2 , 4)
   z.n       = point: new (4 , 4)
   L.ab      = line : new (z.a,z.b)
   L.mn      = line : new (z.m,z.n)
   z.c       = L.ab. mid
   z.e       = L.ab: point (-.2)
   E         = ellipse: foci (z.a,z.b,z.e)
   z.u,z.v   = intersection (E,L.mn)
  -- transfer to tex
   a         = E.Rx
   b         = E.Ry
   ang       = math.deg(E.slope)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines[red](a,b u,v) % p,s p,t
   \tkzDrawPoints(a,b,c,e,u,v) %
   \tkzLabelPoints(a,b,c,u,v)
   \tkzDrawEllipse[teal](c,\tkzUseLua{a},\tkzUseLua{b},\tkzUseLua{ang})
   \tkzDrawSegments(c,u c,v)
   \tkzFillAngles[green!30,opacity=.4](e,c,v)
   \tkzFillAngles[green!80,opacity=.4](e,c,u)
\end{tikzpicture}
```
Other examples: 12.2.2, 23.32

22 In-depth study

22.1 The tables

22.1.1 General tables

Tables are the only data structure ”container” integrated in Lua. They are associative arrays which associates a key (reference or index) with a value in the form of a field (set) of key/value pairs. Moreover, tables have no fixed size and can grow based on our need dynamically.
Tables are created using table constructors, the simplest of which is the use of braces, e.g. { }. This defines an empty table.

```
F = {"banana", "apple", "cherry"}
```

print(F[2]) –> pomme
qui peut être également définit par

```
 FR = {[1] = "banana", [3] = "cherry", [2] = "apple"}
```

print(FR[3]) –> cherry
FR[4]=”orange”

```
 print(#FR)
 -- I for Index
 for I,V in ipairs(FR) do
    print(I,V)
 end
```

1 banana
2 apple
3 cherry
4 orange

```
C = {["banana"] = "yellow" , ["apple"] = "green" , ["cherry"] = "red" }
C.orange = "orange"
```

```
 for K,V in pairs (C) do
    print(K,V)
 end
```

banana = yellow cherry = red orange = orange apple = green
Another useful feature is the ability to create a table to store an unknown number of function parameters, for example:

```
function ReturnTable (...)
  return table.pack (...)
end
```

```
function ParamToTable (...)
  mytab =  ReturnTable(...)
    for i=1,mytab.n do
      print(mytab[i])
    end
end
ParamToTable("cherry","apple","orange")
```

Using tables with table[key] syntax:
`C["banana"] and F[1]`
But with string constants as keys we have the sugar syntax: C.banana but this syntax does not accept numbers.
It's possible to erase a key/value pair from a table, with :

```
C.banana = nil
```

22.1.2 Table z

This is the most important table in the package. It stores all points and enables them to be transferred to Ti*k*Z. It is defined with `z = {}`, then each time we write

```
z.name = point : new (a , b)
```

a point object is stored in the table. The key is `name`, the value is an object. We have seen that `z.name.re = a` and that `z.name.im = b`.
However, the elements of this table have essential properties.
For example, if you wish to display an element, then `tex.print(tostring(z.name)) = a+ib` the `tostring` operation displays the affix corresponding to the point.
In addition, we'll see that it's possible to perform operations with the elements of the `z` table.

22.2 Transfers

We've seen (sous-section 6.1.1) that the macro transfers point coordinates to Ti*k*Z. Let's take a closer look at this macro:

```
\def\tkzGetNodes{\directlua{%
   for K,V in pairs(z) do
      local K,n,sd,ft
      n = string.len(KS)
      if n >1 then
      _,_,ft, sd = string.find( K , "(.+)(.)" )
     if sd == "p" then   K=ft.."'" end
       end
  tex.print("\\coordinate ("..K..") at ("..V.re..","..V.im..") ;\\\\")
end}
}
```

It consists mainly of a loop. The variables used are K (for keys) and V (for Values). To take pairs (key/value) from the `z` table, use the `pairs` function. K becomes the name of a node whose coordinates are `V.re` and `V.im`. Meanwhile, we search for keys with more than one symbol ending in `p`, in order to associate them with the symbol "'" valid in Ti*k*Z.

22.3 Complex numbers library and point

Unless you want to create your own functions, you won't need to know and use complex numbers. However, in some cases it may be useful to implement some of their properties.
`z.A =  point : new (1,2 )` and `z.B = point : new (1,-1)` define two affixes which are $z_A = 1+2i$ and $z_B = 1-i$. Note the difference in notations `z.A` and z_A for two distinct entities: a Lua object and an affix.

If you want to use only complex numbers then you must choose the following syntax :`za =point (1,2)`. The difference between `z.A = point : new (1,2)` and `za = point (1,2)` is that the first function takes into account the scale. If `scale = 2` then $z_A = 2+4i$. In addition, the object referenced by A is stored in table z and not za.
The notation may come as a surprise, as I used the term "point". The aim here was not to create a complete library on complex numbers, but to be able to use their main properties in relation to points. I didn't want to have two different levels, and since a unique connection can be established between the points of the plane and the complexes, I decided not to mention the complex numbers! But they are there.

Table 29: Point or complex metamethods.

Metamethods	**Application**	
`__add(z1,z2)`	`z.a + z.b`	affix
`__sub(z1,z2)`	`z.a - z.b`	affix
`__unm(z)`	`- z.a`	affix
`__mul(z1,z2)`	`z.a * z.b`	affix
`__concat(z1,z2)`	`z.a .. z.b`	dot product = real number [a]
`__pow(z1,z2)`	`z.a ^ z.b`	determinant = real number
`__div(z1,z2)`	`z.a / z.b`	affix
`__tostring(z)`	tex.print(tostring(z))	displays the affix
`__tonumber(z)`	tonumber(z)	affix or nil
`__eq(z1,z2)`	eq (z.a,z.b)	boolean

a If O is the origin of the complex plan, then we get the dot product of the vectors $\overrightarrow{Oa}$ and $\overrightarrow{Ob}$

Table 30: Point (complex) class methods.

Methods	**Application**	
`conj(z)`	`z.a : conj()`	affix (conjugate)
`mod(z)`	`z.a : mod()`	real number = modulus `z.a`
`abs (z)`	`z.a : abs()`	real number = modulus
`norm (z)`	`z.a : norm()`	norm (real number)
`arg (z)`	`z.a : arg()`	real number = argument of z.a (in rad)
`get(z)`	`z.a : get()`	re and im (two real numbers)
`sqrt(z)`	`z.a : sqrt()`	affix

The class is provided with two specific metamethods.

– Since concatenation makes little sense here, the operation associated with `..` is the scalar or dot product. If `z1 = a+ib` and `z2 = c+id` then

  `z1..z2 = (a+ib) .. (c+id) = (a+ib) (c-id) = ac+bd + i(bc-ad)`

  There's also a mathematical function, `dot_product`, which takes three arguments. See example 20.9

– With the same idea, the operation associated with `^` is the determinant i.e.

  `z1 ^ z2 = (a+ib) ^ (c+id) = ad - bc  From  (a-ib) (c+id) = ac+bd + i(ad - bc)` we take the imaginary part.

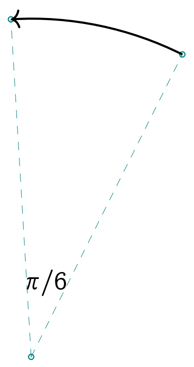
22.3.1 Example of complex use

Let `za = math.cos(a) + i math.sin(a)` . This is obtained from the library by writing

```
za = point(math.cos(a),math.sin(a)).
```

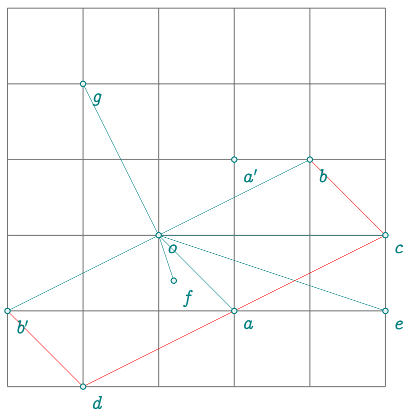
Then `z.B = z.A * za` describes a rotation of point A by an angle a.

```
\begin{tkzelements}
   z.O = point : new (0,0)
   z.A = point : new (1,2)
   a = math.pi/6
   za = point(math.cos(a),math.sin(a))
   z.B = z.A * za
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPoints(O,A,B)
\tkzDrawArc[->,delta=0](O,A)(B)
\tkzDrawSegments[dashed](O,A O,B)
\tkzLabelAngle(A,O,B){$\pi/6$}
\end{tikzpicture}
```



22.3.2 Point operations(complex)

```
\begin{tkzelements}
   z.o  = point: new(0,0)
   z.a  = point: new(1,-1)
   z.b  = point: new(2,1)
   z.bp = -z.b
   z.c  = z.a + z.b
   z.d  = z.a - z.b
   z.e  = z.a * z.b
    z.f  = z.a / z.b
    z.ap = point.conj (z.a)
    -- = z.a : conj ()
   z.g = z.b* point(math.cos(math.pi/2),
                    math.sin(math.pi/2))
\end{tkzelements}

\hspace*{\fill}
\begin{tikzpicture}
 \tkzGetNodes
 \tkzInit[xmin=-2,xmax=3,ymin=-2,ymax=3]
 \tkzGrid
 \tkzDrawSegments(o,a o,b o,c o,e o,b' o,f o,g)
 \tkzDrawSegments[red](a,c b,c b',d a,d)
 \tkzDrawPoints(a,...,g,o,a',b')
 \tkzLabelPoints(o,a,b,c,d,e,f,g,a',b')
\end{tikzpicture}
```



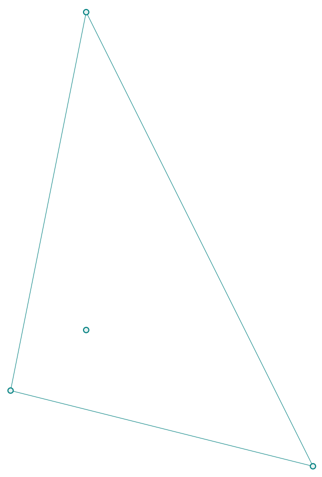
22.4 Barycenter

Here's the definition of the barycenter, which is used some forty times in the package.
`table.pack` builds a table from a list.
`tp.n` gives the number of pairs.
`tp[i][1]` is an affix and `tp[i][2]` the associated weight (real value). 5se the example.

```
function barycenter_ (...)
local tp = table.pack(...)
local i
local sum = 0
local weight=0
for i=1,tp.n do
   sum = sum + tp[i][1]*tp[i][2]
   weight = weight + tp[i][2]
end
return sum/weight
end
```

22.4.1 Using the barycentre

```
\begin{tkzelements}
 z.A =  point: new (1,0)
 z.B =  point: new (5,-1)
 z.C =  point: new (2,5)
 z.G =  barycenter ({z.A,3},{z.B,1},{z.C,1})
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C)
\tkzDrawPoints(A,B,C,G)
\end{tikzpicture}
```



22.4.2 Incenter of a triangle

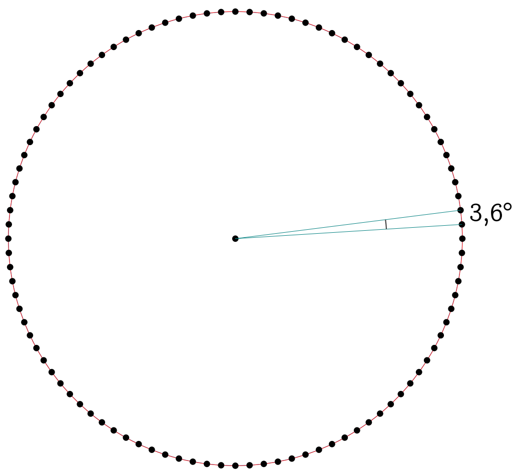
The calculation of the weights ka, kb and kc is precise, and the result obtained with the barycenter is excellent. Note the presence of the underscore _ for certain functions. These functions are internal (developer). Each external (user) function is associated with its internal counterpart.
Here's how to determine the center of the inscribed circle of a triangle:

```
function in_center_ ( a,b,c )
   local ka = point.abs (b-c)
   local kc = point.abs (b-a)
   local kb = point.abs (c-a)
   return    barycenter_ ( {a,ka} , {b,kb} , {c,kc} )
```

22.5 Loop and table notation

The problem encountered in this example stems from the notation of the point names. Since it's not possible to write in simplified form, we have to resort to table[key] notation.

```
\begin{tkzelements}
  local r   = 3
  z.O       = point : new (0,0)
  max       = 100
  for i     = 1,max
  do
     z["A_"..i] = point : polar(r,2*i*math.pi/max)
  end
  a = math.deg(get_angle (z.O,z.A_1,z.A_2))
\end{tkzelements}
```



22.6 In_out method

This function can be used for the following objects

- line
- circle
- triangle
- ellipse

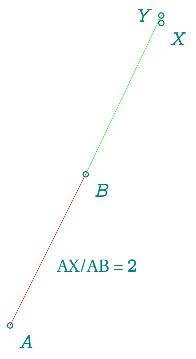
The disk object doesn't exist, so with `in\_out\_disk` it's possible to determine whether a point is in a disk.

22.6.1 In_out for a line

```
function line: in_out (pt)
local sc,epsilon
epsilon = 10^(-12)
sc = math.abs ((pt-self.pa)^(pt-self.pb))
if sc <= epsilon
   then
      return true
   else
      return false
   end
end
```

The `ifthen` package is required for the code below.

```
\begin{tkzelements}
z.A     = point: new (0,0)
z.B     = point: new (1,2)
z.X     = point: new (2,4.000)
z.Y     = point: new (2,4.1)
L.AB = line :  new (z.A,z.B)
if L.AB : in_out (z.X)
  then
   inline = true  k = (z.X-z.A)/(z.B-z.A)
  else
   inline = false
  end
 inline_bis = L.AB : in_out (z.Y)
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPoints(A,B,X,Y)
\tkzLabelPoints(A,B,X)
\tkzLabelPoints[left](Y)
\ifthenelse{\equal{\tkzUseLua{inline}}{true}}{
   \tkzDrawSegment[red](A,B)
   \tkzLabelSegment(A,B){AX/AB = $\tkzUseLua{k}$}}{%
   \tkzDrawSegment[blue](A,B)}
\ifthenelse{\equal{\tkzUseLua{inline_bis}}{false}}{%
 \tkzDrawSegment[green](B,Y)}{}
\end{tikzpicture}
```



22.7 Determinant and dot product

22.7.1 Determinant

We've just seen how to use ^ to obtain the determinant associated with two vectors.
in_out is simply a copy of islinear .
Here's the definition and transformation of the power of a complex number.

```
   -- determinant  is '^'   ad - bc
   function point.__pow(z1,z2)
       local z
       z = point.conj(z1) * z2   -- (a-ib) (c+id) = ac+bd + i(ad - bc)
      return z.im
   end
```

22.7.2 Dot product

Here's the definition of the dot product between two affixes and the concatenation transformation.

```
-- dot product is '..'         result ac + bd
function point.__concat(z1,z2)
    local z
    z = z1 * point.conj(z2)         -- (a+ib) (c-id) = ac+bd + i(bc-ad)
  return z.re
end
```

22.7.3 Dot product: orthogonality test

Here’s a function `isortho` to test orthogonality between two vectors.

```
function isortho (z1,z2,z3)
   local epsilon
   local dp
   epsilon = 10^(-8)
   dp = (z2-z1) .. (z3-z1)
   if math.abs(dp) < epsilon
    then
        return true
    else
        return false
    end
end
```

22.7.4 Dot product: projection

The projection of a point onto a straight line is a fundamental function, and its definition is as follows:

```
function projection_ ( pa,pb,pt )
   local v
   local z
   if aligned ( pa,pb,pt ) then
   return pt
   else
    v = pb - pa
    z = ((pt - pa)..v)/(point.norm(v)) -- .. dot product
   return pa + z * v
   end
end
```

The function `aligned` is equivalent to `islinear` but does not use a determinant. It will be replaced in a future version.

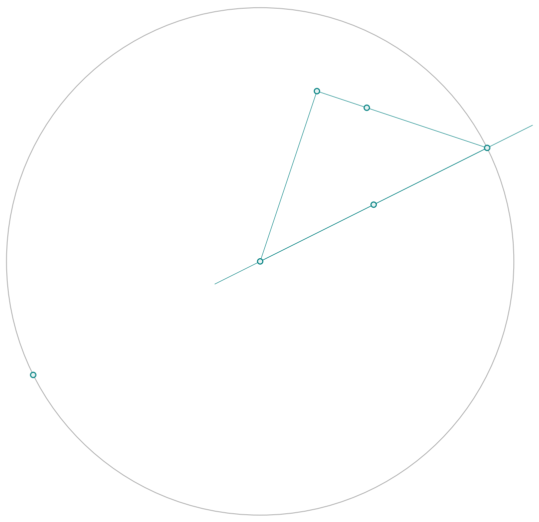
22.8 Point method

The point method is a method for many objects:

- line ,
- circle,
- ellipse,
- triangle.

You obtain a point on the object by entering a real number between 0 and 1.

```
\begin{tkzelements}
   z.A   = point : new ( 0 , 0 )
   z.B   = point : new ( 4 , 2 )
   z.C   = point : new ( 1 , 3 )
   L.AB  = line : new (z.A,z.B)
   C.AB  = circle  : new (z.A,z.B)
   T.ABC = triangle  : new  (z.A,z.B,z.C)
   z.I   = L.AB : point (0.5)
   z.J   = C.AB : point (0.5)
   z.K   = T.ABC : point (0.5)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLine(A,B)
   \tkzDrawCircle(A,B)
   \tkzDrawPolygon(A,B,C)
   \tkzDrawPoints(A,B,C,I,J,K)
\end{tikzpicture}
```



22.9 Behind the objects

Before introducing objects, I only used functions whose parameters were points (comlexes).
For example, `z.m = midpoint_ (z.a,z.b)` defines the midpoint of points a and b. With objects, first define the line/sgment `L.ab` and then obtain the middle with `z.m = L.ab.mid`.
I've kept the functions (which I'll call "primary") whose only arguments are points. They are distinguished from the others by a terminal underscore. In fact, all (almost) object-related functions depend on a primary function. We've just seen the case of the midpoint of a point, so let's look at two other cases:

– Rotation around a point. `c` is the center of rotation, `a` the angle and `pt` the point to be affected. For example: `z.Mp = rotation (z.A,math.pi/6,z.M)`

```
function rotation_ (c,a,pt)
local z = point( math.cos(a) , math.sin(a) )
return z*(pt-c)+c
end
```

  With objects, this gives `z.Mp = z.A : rotation (math.pi/6,z.M)`

– The intersection of a line and a circle is obtained using `intersection_lc_ (z.A,z.B,z.O,z.T)`. using the straight line (A,B) and the circle $C(O,T)$.

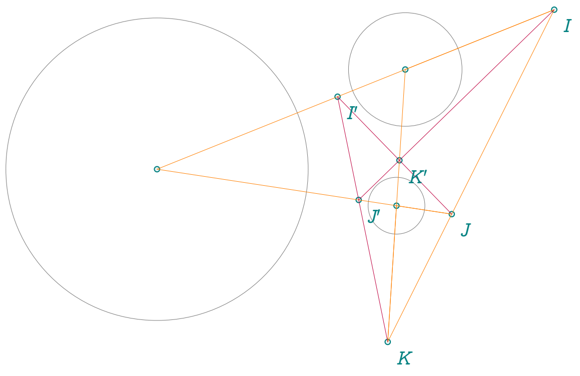
  This will result in the objects: `intersection (L.AB,C.OT)`

The difference is that programming is more direct with primary functions and a little more efficient, but loses visibility.

23 Examples

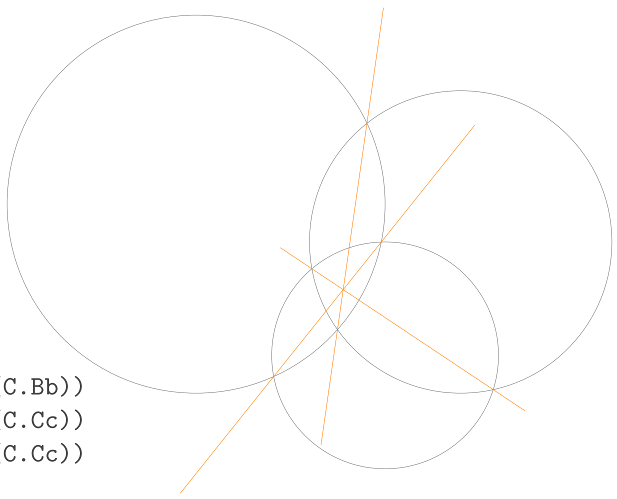
23.1 D'Alembert 1

```
\begin{tkzelements}
   z.A = point : new (0,0)
   z.a = point : new (4,0)
   z.B = point : new (7,-1)
   z.b = point : new (5.5,-1)
   z.C = point : new (5,-4)
   z.c = point : new (4.25,-4)
   C.Aa    = circle :    new (z.A,z.a)
   C.Bb    = circle :    new (z.B,z.b)
   C.Cc    = circle :    new (z.C,z.c)
   z.I = C.Aa : external_similitude (C.Bb)
   z.J = C.Aa : external_similitude (C.Cc)
   z.K = C.Cc : external_similitude (C.Bb)
   z.Ip    = C.Aa : internal_similitude (C.Bb)
   z.Jp    = C.Aa : internal_similitude (C.Cc)
   z.Kp    = C.Cc : internal_similitude (C.Bb)
\end{tkzelements}
\begin{tikzpicture}[rotate=-60]
   \tkzGetNodes
   \tkzDrawCircles(A,a B,b C,c)
   \tkzDrawPoints(A,B,C,I,J,K,I',J',K')
   \tkzDrawSegments[new](I,K A,I A,J B,I B,K C,J C,K)
   \tkzDrawSegments[purple](I,J' I',J I',K)
   \tkzLabelPoints(I,J,K,I',J',K')
\end{tikzpicture}
```

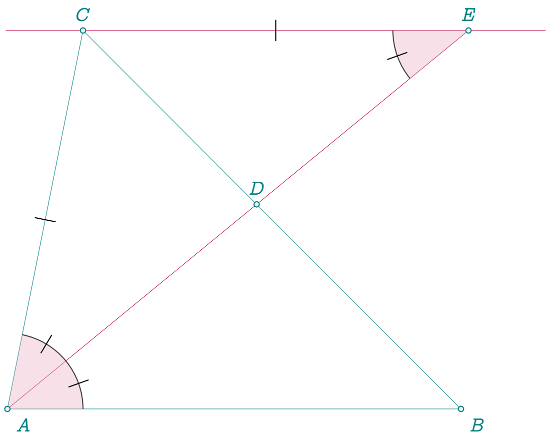


23.2 D'Alembert 2

```
\begin{tkzelements}
   scale = .75
   z.A     = point : new (0,0)
   z.a     = point : new (5,0)
   z.B     = point : new (7,-1)
   z.b     = point : new (3,-1)
   z.C     = point : new (5,-4)
   z.c     = point : new (2,-4)
   C.Aa    = circle :    new (z.A,z.a)
   C.Bb    = circle :    new (z.B,z.b)
   C.Cc    = circle :    new (z.C,z.c)
   z.i,z.j = get_points (C.Aa : radical_axis (C.Bb))
   z.k,z.l = get_points (C.Aa : radical_axis (C.Cc))
   z.m,z.n = get_points (C.Bb : radical_axis (C.Cc))
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,a B,b C,c)
   \tkzDrawLines[new](i,j k,l m,n)
\end{tikzpicture}
```

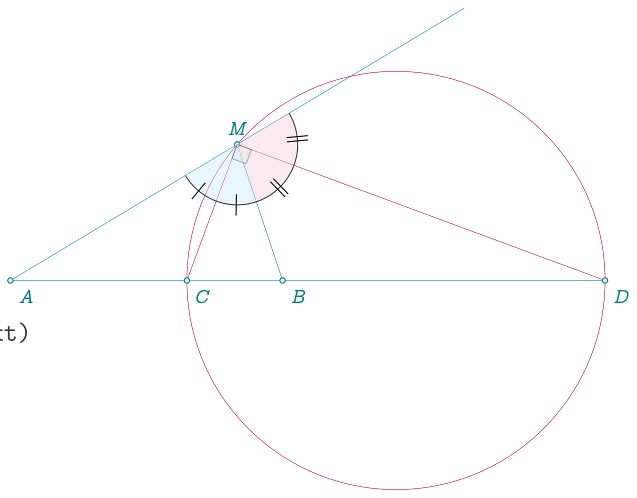


23.3 Alternate



```
\begin{tkzelements}
   z.A  = point: new (0 , 0)
   z.B  = point: new (6 , 0)
   z.C  = point: new (1 , 5)
   T    = triangle: new (z.A,z.B,z.C)
   z.I  = T.incenter
   L.AI = line: new (z.A,z.I)
   z.D  = intersection (L.AI,T.bc)
   L.LLC    = T.ab: ll_from (z.C)
   z.E  = intersection (L.AI,L.LLC)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawLine[purple](C,E)
   \tkzDrawSegment[purple](A,E)
   \tkzFillAngles[purple!30,opacity=.4](B,A,C C,E,D)
   \tkzMarkAngles[mark=|](B,A,D D,A,C C,E,D)
   \tkzDrawPoints(A,...,E)
   \tkzLabelPoints(A,B)
   \tkzLabelPoints[above](C,D,E)
   \tkzMarkSegments(A,C C,E)
\end{tikzpicture}
```

23.4 Apollonius circle

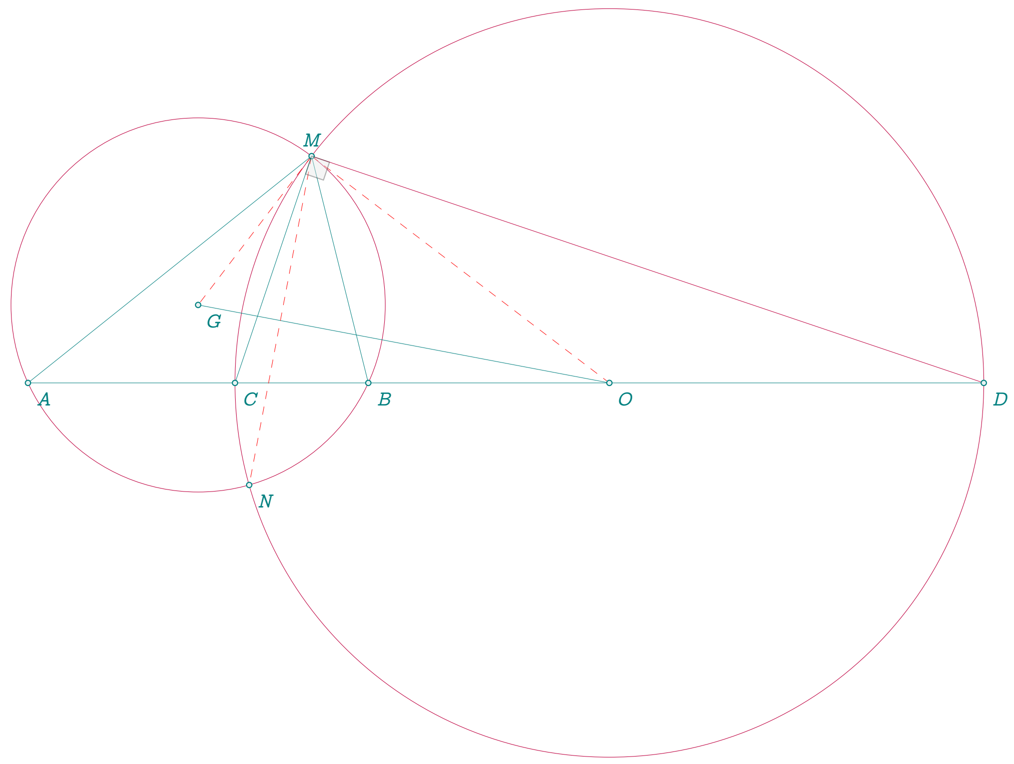


```
\begin{tkzelements}
scale=.75
   z.A      = point: new (0 , 0)
   z.B      = point: new (6 , 0)
   z.M      = point: new (5 , 3)
   T.MAB    = triangle : new (z.M,z.A,z.B)
   L.bis    = T.MAB : bisector ()
   z.C      = L.bis.pb
   L.bisext = T.MAB : bisector_ext ()
   z.D      = intersection (T.MAB.bc, L.bisext)
   L.CD     = line: new (z.C,z.D)
   z.O      = L.CD.mid
   L.AM     = T.MAB.ab
   z.E      = z.M : symmetry (z.A)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawSegment[add=0 and 1](A,M)
   \tkzDrawSegments[purple](M,C M,D)
   \tkzDrawCircle[purple](O,C)
   \tkzDrawSegments(A,B B,M D,B)
   \tkzDrawPoints(A,B,M,C,D)
   \tkzLabelPoints[below right](A,B,C,D)
   \tkzLabelPoints[above](M)
   \tkzFillAngles[opacity=.4,cyan!20](A,M,B)
   \tkzFillAngles[opacity=.4,purple!20](B,M,E)
   \tkzMarkRightAngle[opacity=.4,fill=gray!20](C,M,D)
   \tkzMarkAngles[mark=|](A,M,C C,M,B)
   \tkzMarkAngles[mark=||](B,M,D D,M,E)
\end{tikzpicture}
```

Remark : The circle can be obtained with:
`C.AB = T.MAB.bc : apollonius (length(z.M,z.A)/length(z.M,z.B))`

23.5 Apollonius and circle circumscribed

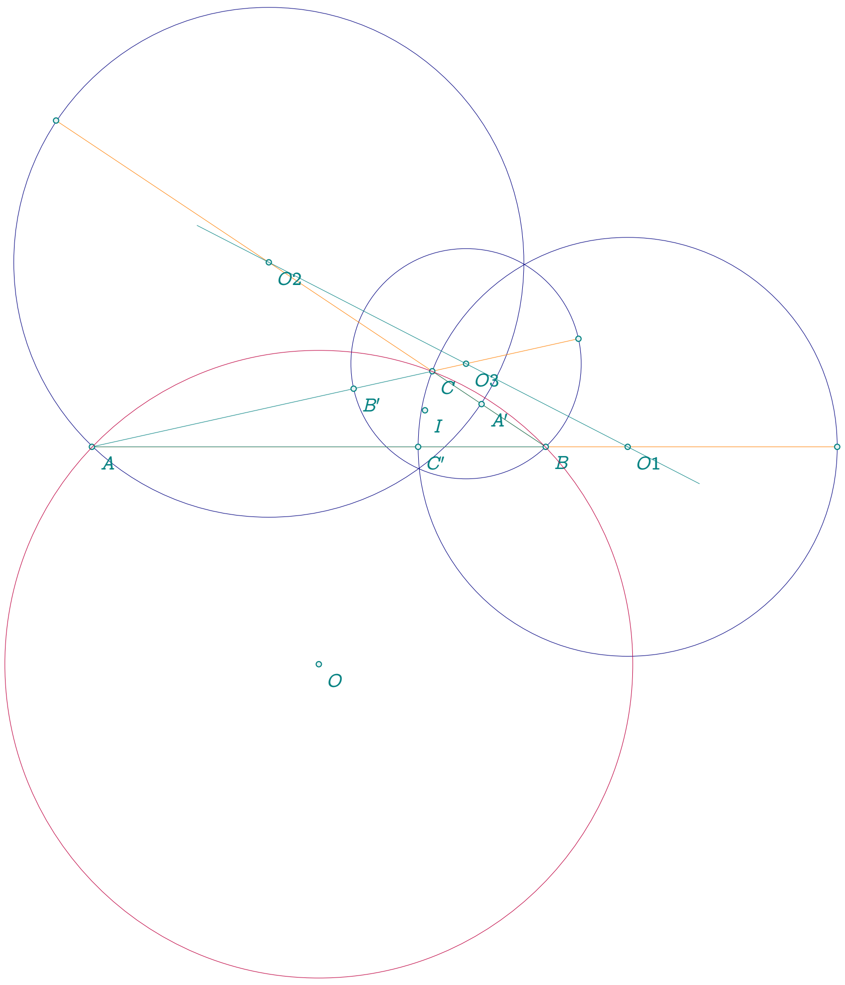
```
\begin{tkzelements}
   scale =.75
   z.A   = point: new (0 , 0)
   z.B   = point: new (6 , 0)
   z.M   = point: new (5 , 4)
   T.AMB = triangle: new (z.A,z.M,z.B)
   L.AB  = T.AMB.ca
   z.I   = T.AMB.incenter
   L.MI  = line: new (z.M,z.I)
   z.C   = intersection (L.AB , L.MI)
   L.MJ  = L.MI: ortho_from (z.M)
   z.D   = intersection (L.AB , L.MJ)
   L.CD  = line: new (z.C,z.D)
   z.O   = L.CD.mid
   z.G   = T.AMB.circumcenter
   C.GA  = circle: new (z.G,z.A)
   C.OC  = circle: new (z.O,z.C)
   _,z.N = intersection (C.GA , C.OC)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,M)
   \tkzDrawCircles[purple](O,C G,A)
   \tkzDrawSegments[purple](M,D)
   \tkzDrawSegments(D,B O,G M,C)
   \tkzDrawSegments[red,dashed](M,N M,O M,G)
   \tkzDrawPoints(A,B,M,C,D,N,O,G)
   \tkzLabelPoints[below right](A,B,C,D,N,O,G)
   \tkzLabelPoints[above](M)
   \tkzMarkRightAngle[opacity=.4,fill=gray!20](C,M,D)
\end{tikzpicture}
```



23.6 Apollonius circles in a triangle

```
\begin{tkzelements}
   z.A   = point: new (0 , 0)
   z.B   = point: new (6 , 0)
   z.C   = point: new (4.5 , 1)
   T.ABC = triangle: new (z.A,z.B,z.C)
   z.I   = T.ABC.incenter
   z.O   = T.ABC.circumcenter
   L.CI  = line: new (z.C,z.I)
   z.Cp  = intersection (T.ABC.ab , L.CI)
   z.x   = L.CI.north_pa
   L.Cx  = line: new (z.C,z.x)
   z.R   = intersection (L.Cx,T.ABC.ab)
   L.CpR = line: new (z.Cp,z.R)
   z.O1  = L.CpR.mid
   L.AI  = line: new (z.A,z.I)
   z.Ap  = intersection (T.ABC.bc , L.AI)
   z.y   = L.AI.north_pa
   L.Ay  = line: new (z.A,z.y)
   z.S   = intersection (L.Ay,T.ABC.bc)
   L.ApS = line: new (z.Ap,z.S)
   z.O2  = L.ApS.mid
   L.BI  = line: new (z.B,z.I)
   z.Bp  = intersection (T.ABC.ca , L.BI)
   z.z   = L.BI.north_pa
   L.Bz  = line: new (z.B,z.z)
   z.T   = intersection (L.Bz,T.ABC.ca)
   L.Bpt = line: new (z.Bp,z.T)
   z.O3  = L.Bpt.mid
```

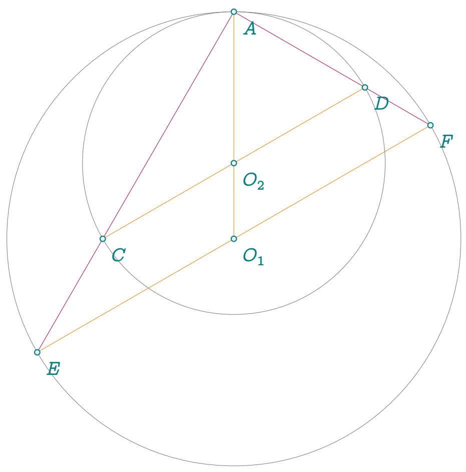
```
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles[blue!50!black](O1,C' O2,A' O3,B')
   \tkzDrawSegments[new](B,S C,T A,R)
   \tkzDrawPolygon(A,B,C)
   \tkzDrawPoints(A,B,C,A',B',C',O,I,R,S,T,O1,O2,O3)
   \tkzLabelPoints(A,B,C,A',B',C',O,I)
   \tkzLabelPoints(O1,O2,O3)
   \tkzDrawCircle[purple](O,A)
   \tkzDrawLine(O1,O2)
\end{tikzpicture}
```



Same result using the function `T.ABC.ab : apollonius (k)`

```
   \begin{tkzelements}
   scale        = .75
   z.A          = point: new (0 , 0)
   z.B          = point: new (6 , 0)
   z.C          = point: new (4.5 , 1)
   T.ABC        = triangle : new (z.A,z.B,z.C)
   z.O          = T.ABC.circumcenter
   C.AB         = T.ABC.ab : apollonius (length(z.C,z.A)/length(z.C,z.B))
   z.w1,z.t1    = get_points ( C.AB )
   C.AC        = T.ABC.ca : apollonius (length(z.B,z.C)/length(z.B,z.A))
   z.w2,z.t2   = get_points ( C.AC )
   C.BC        = T.ABC.bc : apollonius (length(z.A,z.B)/length(z.A,z.C))
   z.w3,z.t3   = get_points ( C.BC )
   \end{tkzelements}
```

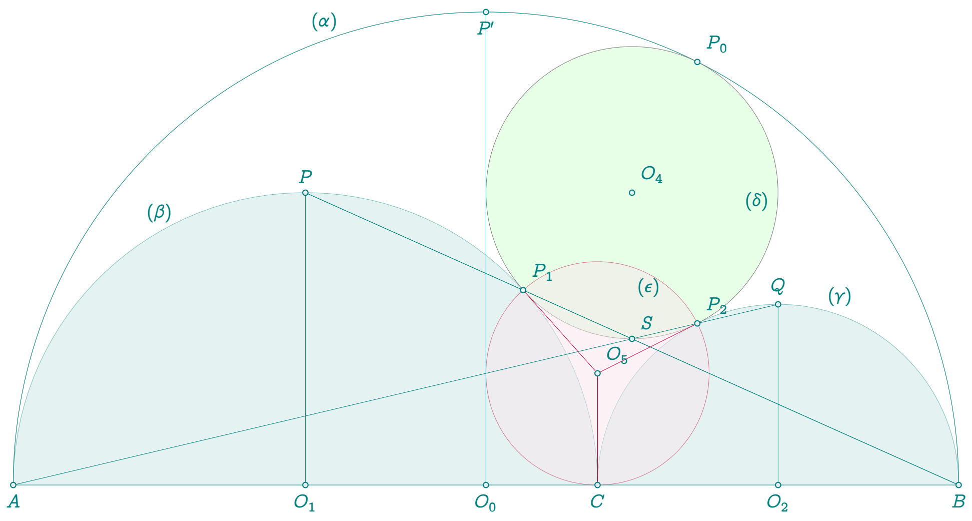
23.7 Archimedes



```
\begin{tkzelements}
   z.O_1     = point:   new  (0, 0)
   z.O_2     = point:   new  (0, 1)
   z.A       = point:   new  (0, 3)
   z.F       = point:   polar (3, math.pi/6)
   L         = line:    new   (z.F,z.O_1)
   C         = circle:  new  (z.O_1,z.A)
   z.E       = intersection (L,C)
   T         = triangle: new (z.F,z.E,z.O_2)
   z.x       = T: parallelogram ()
   L         = line: new    (z.x,z.O_2)
   C         = circle: new  (z.O_2,z.A)
   z.C,z.D   = intersection (L ,C)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O_1,A O_2,A)
   \tkzDrawSegments[new](O_1,A E,F C,D)
   \tkzDrawSegments[purple](A,E A,F)
   \tkzDrawPoints(A,O_1,O_2,E,F,C,D)
   \tkzLabelPoints(A,O_1,O_2,E,F,C,D)
\end{tikzpicture}
```

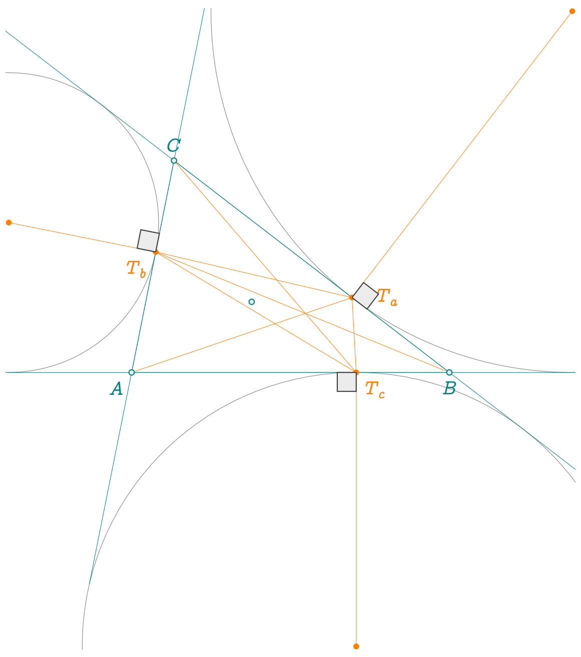
23.8 Bankoff circle

```
\begin{tkzelements}
   z.A       = point: new (0 , 0)
   z.B       = point: new (10 , 0)
   L.AB      = line : new (z.A,z.B)
   z.C       = L.AB: gold_ratio ()
   L.AC      = line : new (z.A,z.C)
   L.CB      = line : new (z.C,z.B)
   z.O_0     = L.AB.mid
   z.O_1     = L.AC.mid
   z.O_2     = L.CB.mid
   C.O0B     = circle : new (z.O_0,z.B)
   C.O1C     = circle : new (z.O_1,z.C)
   C.O2C     = circle : new (z.O_2,z.B)
   z.Pp      = C.O0B : midarc (z.B,z.A)
   z.P       = C.O1C : midarc (z.C,z.A)
   z.Q       = C.O2C : midarc (z.B,z.C)
   L.O1O2    = line : new (z.O_1,z.O_2)
   L.O0O1    = line : new (z.O_0,z.O_1)
   L.O0O2    = line : new (z.O_0,z.O_2)
   z.M_0     = L.O1O2 : harmonic_ext (z.C)
   z.M_1    = L.O0O1 : harmonic_int (z.A)
   z.M_2    = L.O0O2 : harmonic_int (z.B)
   L.BP     = line : new (z.B,z.P)
   L.AQ     = line : new (z.A,z.Q)
   z.S      = intersection (L.BP,L.AQ)
   L.PpO0   = line : new (z.Pp,z.O_0)
   L.PC     = line : new (z.P,z.C)
   z.Ap     = intersection (L.PpO0,L.PC)
   L.CS     = line : new (z.C,z.S)
   C.M1A    = circle : new (z.M_1,z.A)
   C.M2B    = circle : new (z.M_2,z.B)
   z.P_0    = intersection (L.CS,C.O0B)
   z.P_1    = intersection (C.M2B,C.O1C)
   z.P_2    = intersection (C.M1A,C.O2C)
   T.P0P1P2 = triangle : new (z.P_0,z.P_1,z.P_2)
   z.O_4    = T.P0P1P2.circumcenter
   T.CP1P2  = triangle : new (z.C,z.P_1,z.P_2)
   z.O_5    = T.CP1P2.circumcenter
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawSemiCircles[teal](O_0,B)
\tkzDrawSemiCircles[teal,fill=teal!20,opacity=.5](O_1,C O_2,B)
\tkzDrawCircle[fill=green!10](O_4,P_0)
\tkzDrawCircle[purple,fill=purple!10,opacity=.5](O_5,C)
\tkzDrawSegments(A,B O_0,P' B,P A,Q)
\tkzDrawSegments(P,B Q,O_2 P,O_1)
\tkzDrawSegments[purple](O_5,P_2 O_5,P_1 O_5,C)
\tkzDrawPoints(A,B,C,P_0,P_2,P_1,O_0,O_1,O_2,O_4,O_5,Q,P,P',S)
\tkzLabelPoints[below](A,B,C,O_0,O_1,O_2,P')
\tkzLabelPoints[above](Q,P)
\tkzLabelPoints[above right](P_0,P_2,P_1,O_5,O_4,S)
\begin{scope}[font=\scriptsize]
  \tkzLabelCircle[above](O_1,C)(120){$(\beta)$}
  \tkzLabelCircle[above](O_2,B)(70){$(\gamma)$}
  \tkzLabelCircle[above](O_0,B)(110){$(\alpha)$}
  \tkzLabelCircle[left](O_4,P_2)(60){$(\delta)$}
  \tkzLabelCircle[left](O_5,C)(140){$(\epsilon)$}
\end{scope}
\end{tikzpicture}
```



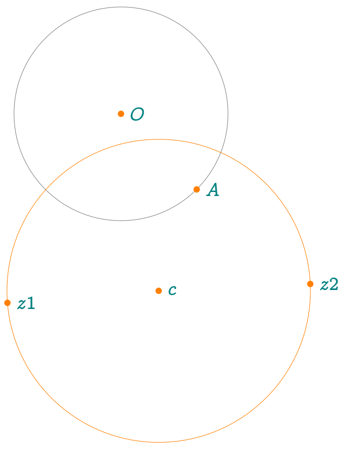
23.9 Excircles

```
\begin{tkzelements}
   scale                  =  0.7
   z.A                    = point: new (0,0)
   z.B                    = point: new (6,0)
   z.C                    = point: new (.8,4)
   T                      = triangle: new ( z.A, z.B, z.C)
   z.K                    = T.centroid
   z.J_a,z.J_b,z.J_c      = get_points (T: excentral())
   z.T_a,z.T_b,z.T_c      = get_points (T: extouch())
   la                     = line: new ( z.A, z.T_a)
   lb                     = line: new ( z.B, z.T_b)
   z.G                    = intersection (la,lb)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPoints[new](J_a,J_b,J_c)
   \tkzClipBB
   \tkzDrawCircles[gray](J_a,T_a J_b,T_b J_c,T_c)
   \tkzDrawLines[add=1 and 1](A,B B,C C,A)
   \tkzDrawSegments[new](A,T_a B,T_b C,T_c)
   \tkzDrawSegments[new](J_a,T_a J_b,T_b J_c,T_c)
   \tkzDrawPolygon(A,B,C)
   \tkzDrawPolygon[new](T_a,T_b,T_c)
   \tkzDrawPoints(A,B,C,K)
   \tkzDrawPoints[new](T_a,T_b,T_c)
   \tkzLabelPoints[below left](A)
   \tkzLabelPoints[below](B)
   \tkzLabelPoints[above](C)
   \tkzLabelPoints[new,below left](T_b)
   \tkzLabelPoints[new,below right](T_c)
   \tkzLabelPoints[new,right=6pt](T_a)
   \tkzMarkRightAngles[fill=gray!15](J_a,T_a,B J_b,T_b,C J_c,T_c,A)
\end{tikzpicture}
```



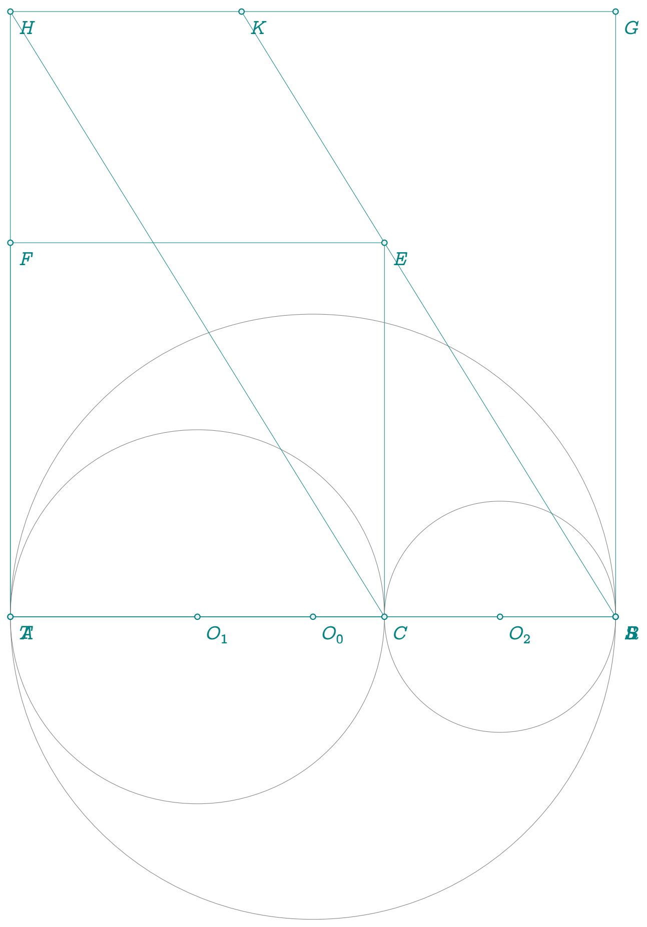
23.10 Orthogonal circle through

```
\begin{tkzelements}
   z.O    = point: new (0,1)
   z.A    = point: new (1,0)
   z.z1   = point: new (-1.5,-1.5)
   z.z2   = point: new (2.5,-1.25)
   C.OA   = circle: new (z.O,z.A)
   C      = C.OA: orthogonal_through (z.z1,z.z2)
   z.c    = C.center
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(O,A)
   \tkzDrawCircle[new](c,z1)
   \tkzDrawPoints[new](O,A,z1,z2,c)
   \tkzLabelPoints[right](O,A,z1,z2,c)
\end{tikzpicture}
```

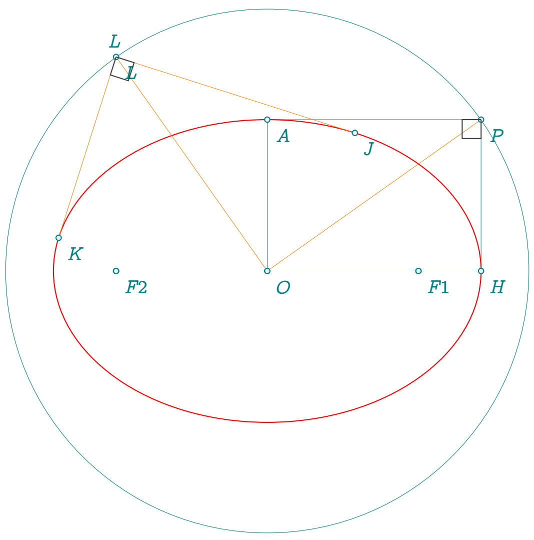


23.11 Divine ratio

```
\begin{tkzelements}
z.A         = point: new (0 , 0)
z.B         = point: new (8 , 0)
L.AB        = line: new (z.A,z.B)
z.C         = L.AB: gold_ratio ()
L.AC        = line: new (z.A,z.C)
z.O_1       = L.AC.mid
_,_,z.G,z.H = get_points(L.AB: square ())
_,_,z.E,z.F = get_points(L.AC: square ())
L.CB        = line: new (z.C,z.B)
z.O_2       = L.CB.mid
z.O_0       = L.AB.mid
L.BE        = line: new (z.B,z.E)
L.GH        = line: new (z.G,z.H)
z.K         = intersection (L.BE,L.GH)
C0          = circle: new (z.O_0,z.B)
z.R,_       = intersection (L.BE,C0)
C2          = circle: new (z.O_2,z.B)
z.S,_       = intersection (L.BE,C2)
L.AR        = line:   new (z.A,z.R)
C1          = circle: new (z.O_1,z.C)
_,z.T       = intersection (L.AR,C1)
L.BG        = line: new (z.B,z.G)
z.L         = intersection (L.AR,L.BG)
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygons(A,C,E,F A,B,G,H)
\tkzDrawCircles(O_1,C O_2,B O_0,B)
\tkzDrawSegments(H,C B,K A,L)
\tkzDrawPoints(A,B,C,K,E,F,G,H,O_0,O_1,O_2,R,S,T,L)
\tkzLabelPoints(A,B,C,K,E,F,G,H,O_0,O_1,O_2,R,S,T,L)
\end{tikzpicture}
```


H
K
G
F
E
T
A
O_1
O_0
C
O_2
S
B

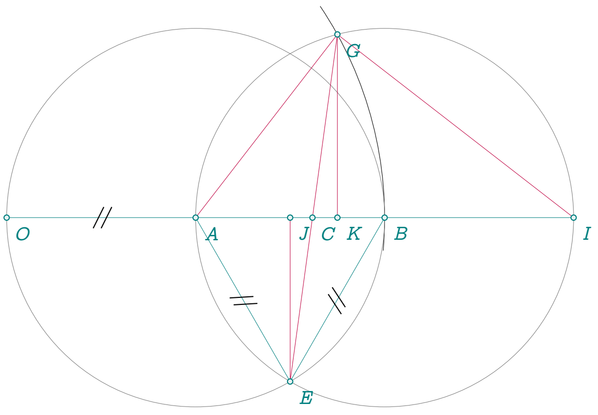
23.12 Director circle



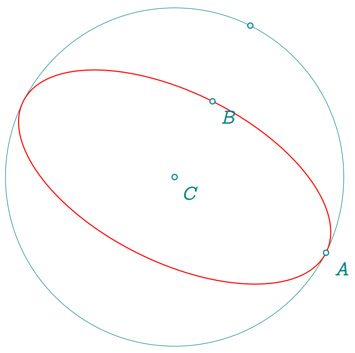
```
\begin{tkzelements}
  scale     = .5
  z.O       = point: new (0 , 0)
  z.F1      = point: new (4 , 0)
  z.F2      = point: new (-4 , 0)
  z.H       = point: new (4*math.sqrt(2) , 0)
  E         = ellipse: foci (z.F2,z.F1,z.H)
  a,b       = E.Rx, E.Ry
  z.A       = E.covertex
  T         = triangle: new (z.H,z.O,z.A)
  z.P       = T: parallelogram ()
  C         = circle: new (z.O,z.P)
  z.L       = C: point (0.25)
  L.J,L.K   = E: tangent_from (z.L)
  z.J       = L.J.pb
  z.K       = L.K.pb
\end{tkzelements}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawPoints(F1,F2,O)
  \tkzDrawCircles[teal](O,P)
  \tkzDrawPolygon(H,O,A,P)
  \tkzDrawEllipse[red](O,\tkzUseLua{a},\tkzUseLua{b},0)
  \tkzDrawSegments[orange](O,P O,L L,J L,K)
  \tkzDrawPoints(F1,F2,O,H,A,P,L,J,K)
  \tkzLabelPoints(F1,F2,O,H,A,P,L,J,K)
  \tkzLabelPoints[above](L)
  \tkzMarkRightAngles(A,P,H J,L,K)
\end{tikzpicture}
```

23.13 Gold division

```
\begin{tkzelements}
z.A          = point:  new (0,0)
z.B          = point:  new (2.5,0)
L.AB         = line:   new (z.A,z.B)
C.AB         = circle: new (z.A,z.B)
C.BA         = circle: new (z.B,z.A)
z.J          = L.AB: midpoint ()
L.JB         = line:new (z.J,z.B)
z.F,z.E      = intersection (C.AB , C.BA)
z.I,_        = intersection (L.AB , C.BA)
z.K          = L.JB : midpoint ()
L.mediator   = L.JB: mediator ()
z.G          = intersection (L.mediator,C.BA)
L.EG         = line:new (z.E,z.G)
z.C          = intersection  (L.EG,L.AB)
z.O          = C.AB: antipode (z.B)
\end{tkzelements}
   \begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawArc[delta=5](O,B)(G)
   \tkzDrawCircles(A,B B,A)
   \tkzDrawSegments(A,E B,E O,I)
   \tkzDrawSegments[purple](J,E A,G G,I K,G E,G)
   \tkzMarkSegments[mark=s||](A,E B,E O,A)
   \tkzDrawPoints(A,B,C,E,I,J,G,O,K)
   \tkzLabelPoints(A,B,C,E,I,J,G,O,K)
   \end{tikzpicture}
```



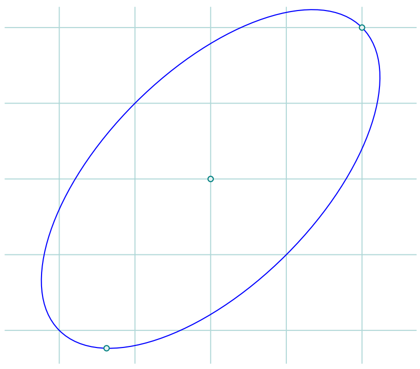
23.14 Ellipse



```
\begin{tkzelements}
   z.C        = point: new (3 , 2)
   z.A        = point: new (5 , 1)
   L.CA       = line : new (z.C,z.A)
   z.b        = L.CA.north_pa
   L          = line : new (z.C,z.b)
   z.B        = L : point (0.5)
   E          = ellipse: new (z.C,z.A,z.B)
   a          = E.Rx
   b          = E.Ry
   slope      = math.deg(E.slope)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles[teal](C,A)
   \tkzDrawEllipse[red](C,\tkzUseLua{a},\tkzUseLua{b},\tkzUseLua{slope})
   \tkzDrawPoints(C,A,B,b)
   \tkzLabelPoints(C,A,B)
\end{tikzpicture}
```

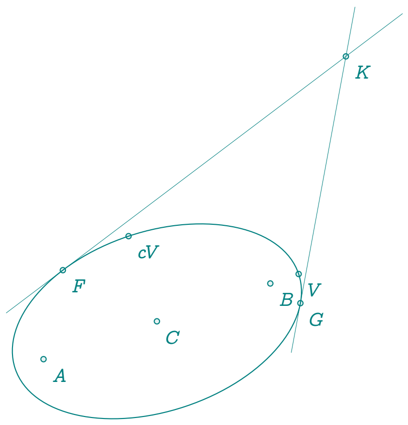
23.15 Ellipse with radii

```
\begin{tkzelements}
scale=.5
z.C    = point: new (0 , 4)
b      = value(math.sqrt(8))
a      = value(math.sqrt(32))
ang    = math.deg(math.pi/4)
E      = ellipse: radii (z.C,a,b,math.pi/4)
z.V    = E : point (0)
z.CoV  = E : point (math.pi/2)
\end{tkzelements}
\begin{tikzpicture}[gridded]
\tkzGetNodes
\tkzDrawEllipse[blue](C,\tkzUseLua{a},
              \tkzUseLua{b},\tkzUseLua{ang})
\tkzDrawPoints(C,V,CoV)
\end{tikzpicture}
```



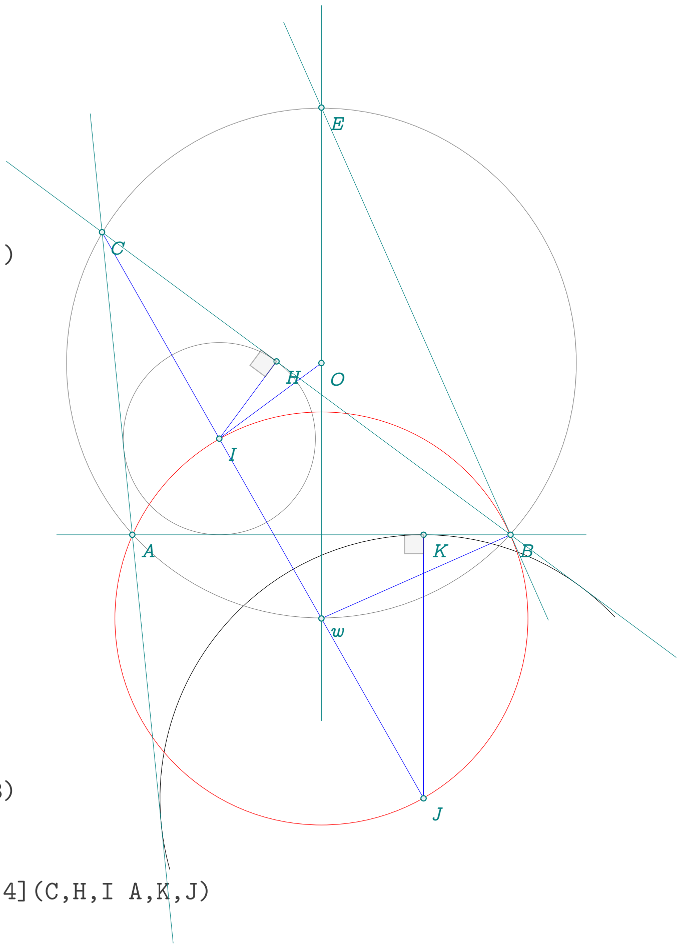
23.16 Ellipse_with_foci

```
\begin{tkzelements}
   local e
   e           = .8
   z.A         = point: new (2 , 3)
   z.B         = point: new (5 , 4)
   z.K         = point: new (6, 7)
   L.AB        = line: new (z.A,z.B)
   z.C         = L.AB.mid
   c           = point.abs(z.B-z.C)
   a           = c/e
   b           = math.sqrt (a^2-c^2)
   z.V         = z.C + a*(z.B-z.C)/point.abs(z.B-z.C)
   E           = ellipse: foci (z.A,z.B,z.V)
   z.cV        = E.covertex
   ang         = math.deg(E.slope)
   L.ta,L.tb   = E: tangent_from (z.K)
   z.F         = L.ta.pb
   z.G         = L.tb.pb
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPoints(A,B,C,K,F,G,V,cV)
   \tkzLabelPoints(A,B,C,K,F,G,V,cV)
   \tkzDrawEllipse[teal](C,\tkzUseLua{a},\tkzUseLua{b},\tkzUseLua{ang})
   \tkzDrawLines(K,F K,G)
\end{tikzpicture}
```

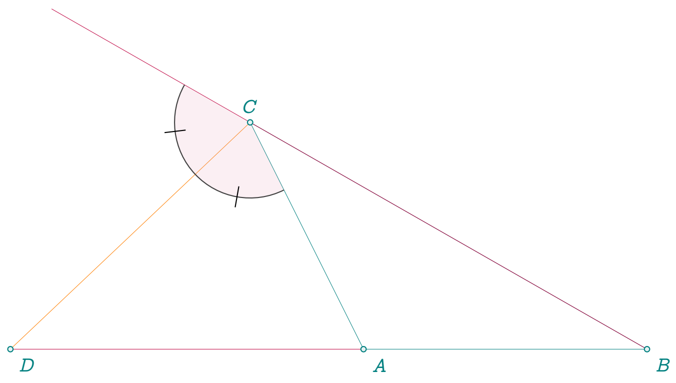


23.17 Euler relation

```
\begin{tkzelements}
 scale      =  .75
 z.A        = point: new (0 , 0)
 z.B        = point: new (5 , 0)
 z.C        = point: new (-.4 , 4)
 T.ABC      = triangle: new (z.A,z.B,z.C)
 z.J,z.K    = get_points(T.ABC: ex_circle (2))
 z.X,z.Y,z.K= T.ABC : projection (z.J)
 z.I,z.H    = get_points(T.ABC : in_circle())
 z.O        = T.ABC.circumcenter
 C.OA       = circle : new (z.O,z.A)
 T.IBA      = triangle: new (z.I,z.B,z.A)
 z.w        = T.IBA.circumcenter
 L.Ow       = line : new (z.O,z.w)
 _,z.E      = intersection (L.Ow, C.OA)
\end{tkzelements}
   \begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawArc(J,X)(Y)
   \tkzDrawCircles(I,H O,A)
   \tkzDrawCircle[red](w,I)
   \tkzDrawLines(Y,C A,B X,C E,w E,B)
   \tkzDrawSegments[blue](J,C J,K I,H I,O w,B)
   \tkzDrawPoints(A,B,C,I,J,E,w,H,K,O)
   \tkzLabelPoints(A,B,C,J,I,w,H,K,E,O)
   \tkzMarkRightAngles[fill=gray!20,opacity=.4](C,H,I A,K,J)
   \end{tikzpicture}
```

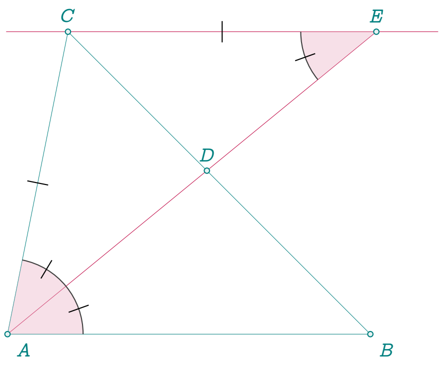


23.18 External angle



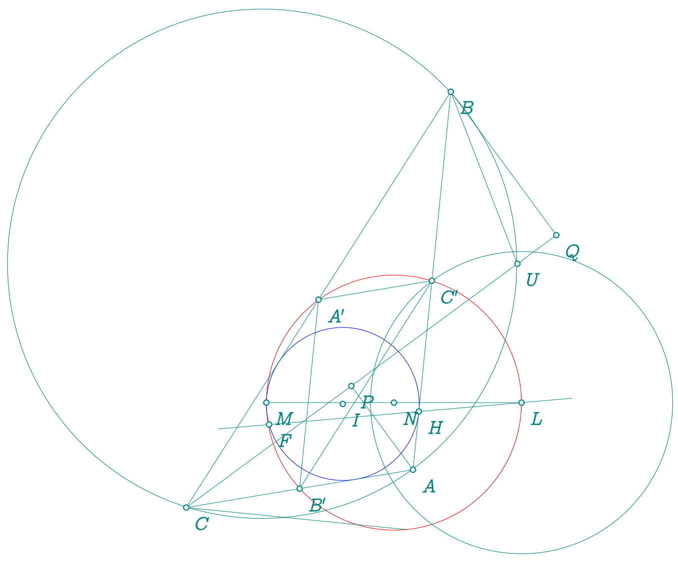
```
\begin{tkzelements}
  scale   = .75
  z.A     = point: new (0 , 0)
  z.B     = point: new (5 , 0)
  z.C     = point: new (-2 , 4)
  T.ABC   = triangle: new (z.A,z.B,z.C)
  T.ext   = T.ABC: excentral ()
  z.O     = T.ABC.circumcenter
  z.D     = intersection (T.ext.ab,T.ABC.ab)
  z.E     = z.C: symmetry (z.B)
\end{tkzelements}
\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawPolygon(A,B,C)
  \tkzDrawLine[purple,add=0 and .5](B,C)
  \tkzDrawSegment[purple](A,D)
  \tkzDrawSegment[orange](C,D)
  \tkzFillAngles[purple!30,opacity=.2](D,C,A E,C,D)
  \tkzMarkAngles[mark=|](D,C,A E,C,D)
  \tkzDrawPoints(A,...,D)
  \tkzLabelPoints[above](C)
  \tkzLabelPoints(A,B,D)
\end{tikzpicture}
```

23.19 Internal angle



```
\begin{tkzelements}
   scale =  .8
   z.A   = point: new (0 , 0)
   z.B   = point: new (6 , 0)
   z.C   = point: new (1 , 5)
   T     = triangle: new (z.A,z.B,z.C)
   z.I   = T.incenter
   L.AI  = line: new (z.A,z.I)
   z.D   = intersection (L.AI, T.bc)
   L.LL  = T.ab: ll_from (z.C)
   L.AD  = line: new (z.A,z.D)
   z.E   = intersection (L.LL,L.AD)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawLine[purple](C,E)
   \tkzDrawSegment[purple](A,E)
   \tkzFillAngles[purple!30,opacity=.4](B,A,C C,E,D)
   \tkzMarkAngles[mark=|](B,A,D D,A,C C,E,D)
   \tkzDrawPoints(A,...,E)
   \tkzLabelPoints(A,B)
   \tkzLabelPoints[above](C,D,E)
   \tkzMarkSegments(A,C C,E)
\end{tikzpicture}
```

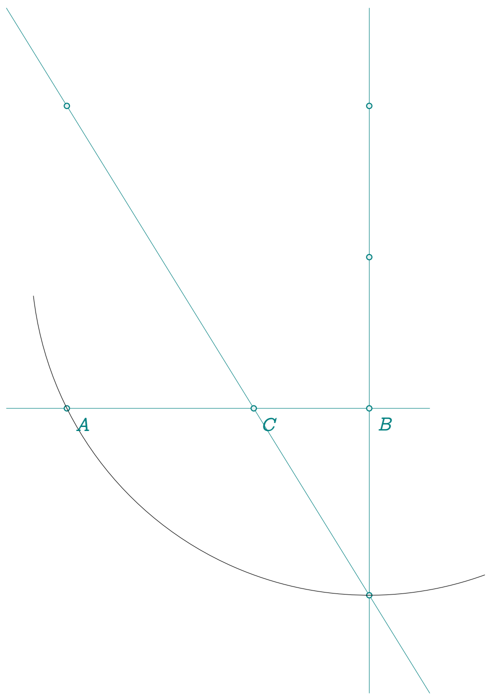
23.20 Feuerbach theorem



```
\begin{tkzelements}
   scale      =  1.5
   z.A        = point: new (0 , 0)
   z.B        = point: new (5 , -.5)
   z.C        = point: new (-.5 , 3)
   T.ABC      = triangle: new (z.A,z.B,z.C)
   z.O        = T.ABC.circumcenter
   z.N        = T.ABC.eulercenter
   z.I,z.K    = get_points(T.ABC: in_circle())
   z.H        = T.ABC.ab : projection (z.I)
   z.Ap,
   z.Bp,
   z.Cp       = get_points (T.ABC : medial ())
   C.IH       = circle:new (z.I,z.H)
   C.NAp      = circle:new (z.N,z.Ap)
   C.OA       = circle:new (z.O,z.A)
   z.U        = C.OA.south
   z.L        = C.NAp.south
   z.M        = C.NAp.north
   z.X        = T.ABC.ab: projection (z.C)
   L.CU       = line: new (z.C,z.U)
   L.ML       = line: new (z.M,z.L)
   z.P        = L.CU: projection (z.A)
   z.Q        = L.CU: projection (z.B)
   L.LH       = line: new (z.L,z.H)
   z.F        = intersection (L.LH,C.IH) -- feuerbach
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLine(L,F)
   \tkzDrawCircle[red](N,A')
   \tkzDrawCircle[blue](I,H)
   \tkzDrawCircles[teal](O,A L,C')
   \tkzDrawSegments(M,L B,U Q,C C,X A,P B,Q)
   \tkzDrawPolygons(A,B,C A',B',C')
   \tkzDrawPoints(A,B,C,N,H,A',B',C',U,L,M,P,Q,F,I)
   \tkzLabelPoints(A,B,C,N,H,A',B',C',U,L,M,P,Q,F,I)
\end{tikzpicture}
```

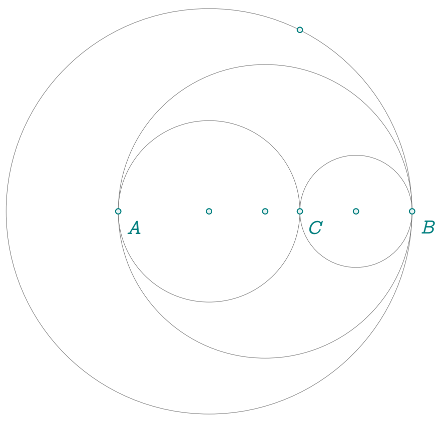
23.21 Gold ratio with segment

```
\begin{tkzelements}
   z.A      = point: new (0 , 0)
   z.B      = point: new (8 , 0)
   L.AB     = line: new (z.A,z.B)
   _,_,z.X,z.Y = get_points(L.AB: square ())
   L.BX     = line: new (z.B,z.X)
   z.M      = L.BX.mid
   C.MA     = circle: new (z.M,z.A)
   _,z.K    = intersection (L.BX,C.MA)
   L.AK     = line: new (z.Y,z.K)
   z.C      = intersection (L.AK,L.AB)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines(A,B X,K)
   \tkzDrawLine[teal](Y,K)
   \tkzDrawPoints(A,B,C,X,Y,M,K)
   \tkzDrawArc[delta=20](M,A)(K)
   \tkzLabelPoints(A,B,C)
\end{tikzpicture}
```

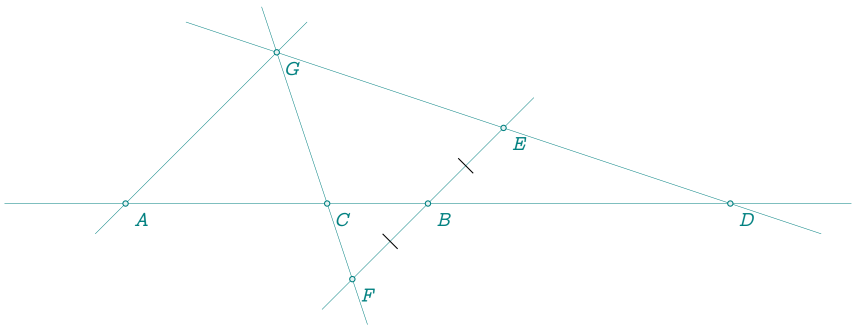


23.22 Gold Arbelos

```
\begin{tkzelements}
   scale    = .6
   z.A      = point: new (0 , 0)
   z.C      = point: new (6 , 0)
   L.AC     = line: new (z.A,z.C)
   _,_,z.x,z.y  = get_points(L.AC: square ())
   z.O_1    = L.AC . mid
   C        = circle: new (z.O_1,z.x)
   z.B      = intersection (L.AC,C)
   L.CB     = line: new (z.C,z.B)
   z.O_2    = L.CB.mid
   L.AB     = line: new (z.A,z.B)
   z.O_0    = L.AB.mid
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O_1,C O_2,B O_0,B)
   \tkzDrawPoints(A,C,B,O_1,O_2,O_0)
   \tkzLabelPoints(A,C,B)
\end{tikzpicture}
```



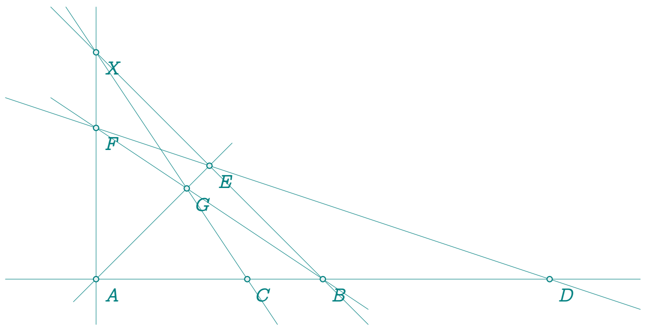
23.23 Harmonic division v1



```
\begin{tkzelements}
   scale=.75
   z.A  = point: new (0 , 0)
   z.B  = point: new (4 , 0)
   z.D  = point: new (12,0)
   L.AB = line : new (z.A,z.B)
   z.X  = L.AB.north_pa
   L.XB = line : new (z.X,z.B)
   z.E  = L.XB.mid
   L.DE = line : new (z.D,z.E)
   L.XA = line : new (z.X,z.A)
   z.F  = intersection (L.DE,L.XA)
   L.AE = line : new (z.A,z.E)
   L.BF = line : new (z.B,z.F)
   z.G  = intersection (L.AE,L.BF)
   L.XG = line : new (z.X,z.G)
z.C  = intersection (L.XG,L.AB)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDefPoints{0/0/A,4/0/B}
   \tkzDefPoints{2/2/G}
   \tkzDefLine[parallel=through B,K=.5](A,G) \tkzGetPoint{E}
   \tkzInterLL(G,E)(A,B)   \tkzGetPoint{D}
   \tkzDefPointBy[symmetry= center B](E) \tkzGetPoint{F}
   \tkzInterLL(G,F)(A,B)   \tkzGetPoint{C}
   \tkzDrawLines(A,D A,G F,E G,F G,D)
   \tkzDrawPoints(A,B,G,E,F,C,D)
   \tkzLabelPoints(A,B,G,E,F,C,D)
   \tkzMarkSegments(F,B B,E)
\end{tikzpicture}
```

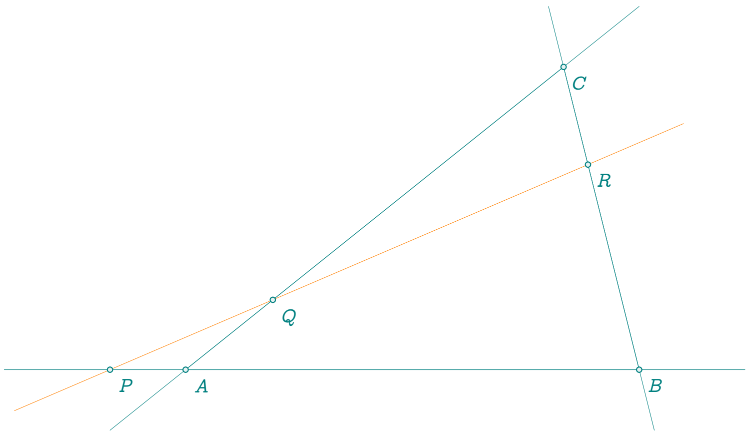
23.24 Harmonic division v2

```
\begin{tkzelements}
   scale    = .5
   z.A      = point: new (0 , 0)
   z.B      = point: new (6 , 0)
   z.D      = point: new (12 , 0)
   L.AB     = line: new (z.A,z.B)
   z.X      = L.AB.north_pa
   L.XB     = line: new (z.X,z.B)
   z.E      = L.XB.mid
   L.ED     = line: new (z.E,z.D)
   L.AX     = line: new (z.A,z.X)
   L.AE     = line: new (z.A,z.E)
   z.F      = intersection (L.ED,L.AX)
   L.BF     = line: new (z.B,z.F)
   z.G      = intersection (L.AE,L.BF)
   L.GX     = line: new (z.G,z.X)
   z.C      = intersection (L.GX,L.AB)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines(A,D A,E B,F D,F X,A X,B X,C)
   \tkzDrawPoints(A,...,G,X)
   \tkzLabelPoints(A,...,G,X)
\end{tikzpicture}
```



23.25 Menelaus

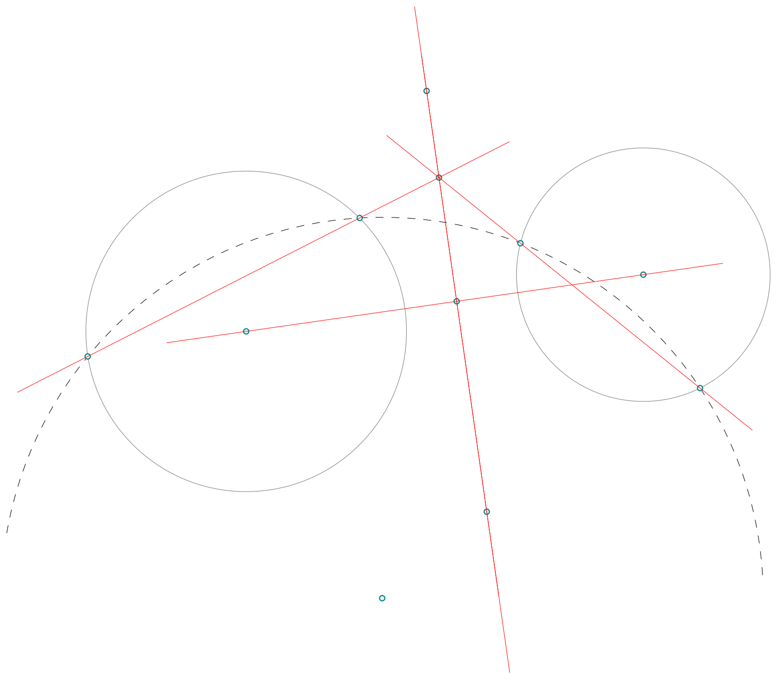
```
\begin{tkzelements}
   z.A  = point: new (0 , 0)
   z.B  = point: new (6 , 0)
   z.C  = point: new (5 , 4)
   z.P  = point: new (-1 , 0)
   z.X  = point: new (6 , 3)
   L.AC = line: new (z.A,z.C)
   L.PX = line: new (z.P,z.X)
   L.BC = line: new (z.B,z.C)
   z.Q  = intersection (L.AC,L.PX)
   z.R  = intersection (L.BC,L.PX)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawLine[new](P,R)
   \tkzDrawLines(P,B A,C B,C)
   \tkzDrawPoints(P,Q,R,A,B,C)
   \tkzLabelPoints(A,B,C,P,Q,R)
\end{tikzpicture}
```



23.26 Radical axis v1

```
\begin{tkzelements}
scale    = .75
```

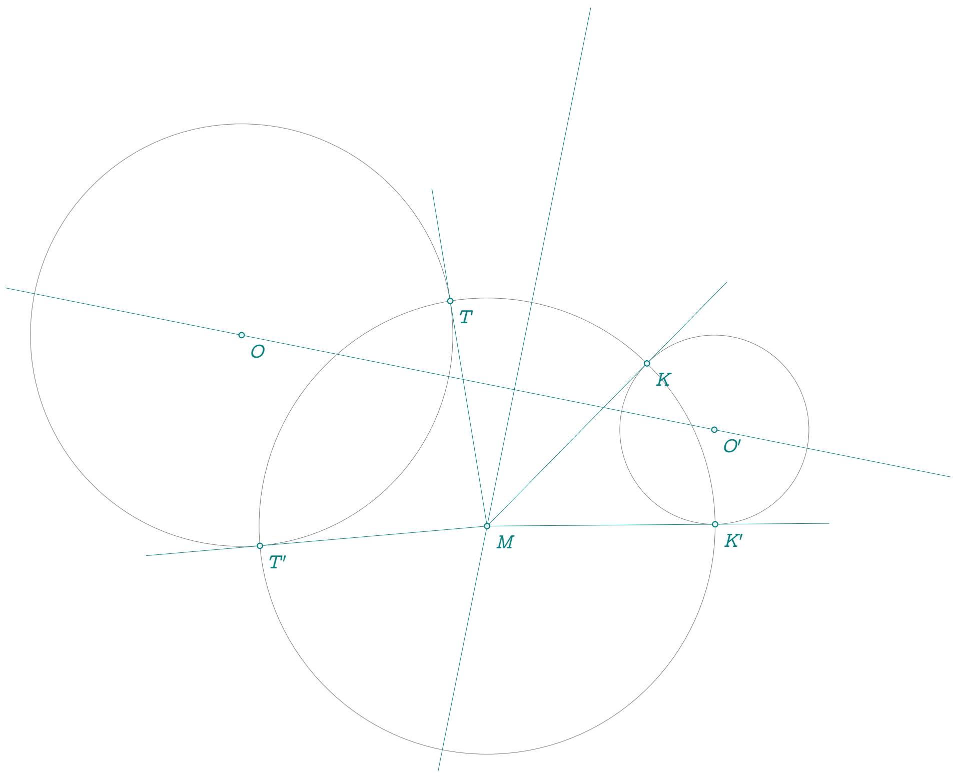
```
z.X       = point : new (0,0)
z.B       = point : new (2,2)
z.Y       = point : new (7,1)
z.Ap      = point : new (8,-1)
L.XY      = line :    new (z.X,z.Y)
C.XB      = circle : new (z.X,z.B)
C.YAp     = circle : new (z.Y,z.Ap)
z.E,z.F   = get_points (C.XB : radical_axis (C.YAp))
z.A       = C.XB : point (0.4)
T.ABAp    = triangle: new (z.A,z.B,z.Ap)
z.O       = T.ABAp.circumcenter
C.OAp     = circle : new (z.O,z.Ap)
_,z.Bp    = intersection (C.OAp,C.YAp)
L.AB      = line : new (z.A,z.B)
L.ApBp    = line : new (z.Ap,z.Bp)
z.M       = intersection (L.AB,L.ApBp)
z.H       = L.XY : projection (z.M)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(X,B Y,A')
   \tkzDrawArc[dashed,delta=30](O,A')(A)
   \tkzDrawPoints(A,B,A',B',M,H,X,Y,O,E,F)
   \tkzDrawLines[red](A,M A',M X,Y E,F)
   \tkzDrawLines[red,add=1 and 3](M,H)
\end{tikzpicture}
```



23.27 Radical axis v2

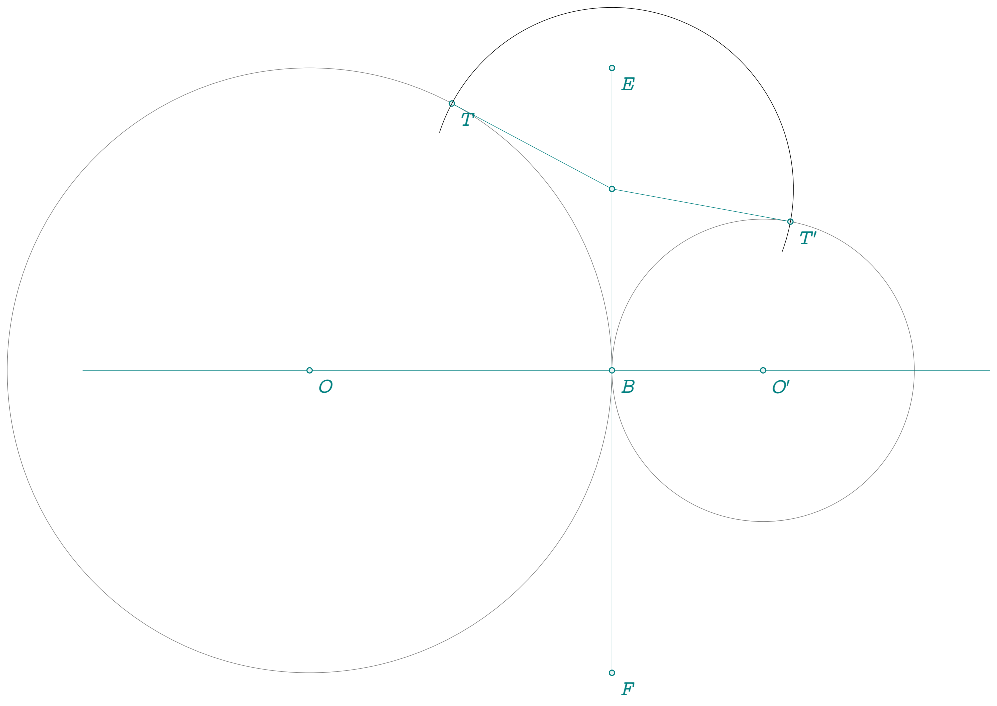
```
\begin{tkzelements}
scale        = 1.25
z.O          = point : new (-1,0)
z.Op         = point : new (4,-1)
```

```
z.B         = point : new (0,2)
z.D         = point : new (4,0)
C.OB        = circle :    new (z.O,z.B)
C.OpD       = circle :    new (z.Op,z.D)
L.EF        = C.OB : radical_axis (C.OpD)
z.E,z.F     = get_points (L.EF)
z.M         = L.EF : point (.75)
L.MT,L.MTp  = C.OB : tangent_from (z.M)
_,z.T       = get_points (L.MT)
_,z.Tp      = get_points (L.MTp)
L.MK,L.MKp  = C.OpD : tangent_from (z.M)
_,z.K       = get_points (L.MK)
_,z.Kp      = get_points (L.MKp)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O,B O',D)
   \tkzDrawLine(E,F)
   \tkzDrawLine[add=.5 and .5](O,O')
   \tkzDrawLines[add = 0 and .5](M,T M,T' M,K M,K')
   \tkzDrawCircle(M,T)
   \tkzDrawPoints(O,O',T,M,T',K,K')
   \tkzLabelPoints(O,O',T,T',K,K',M)
\end{tikzpicture}
```



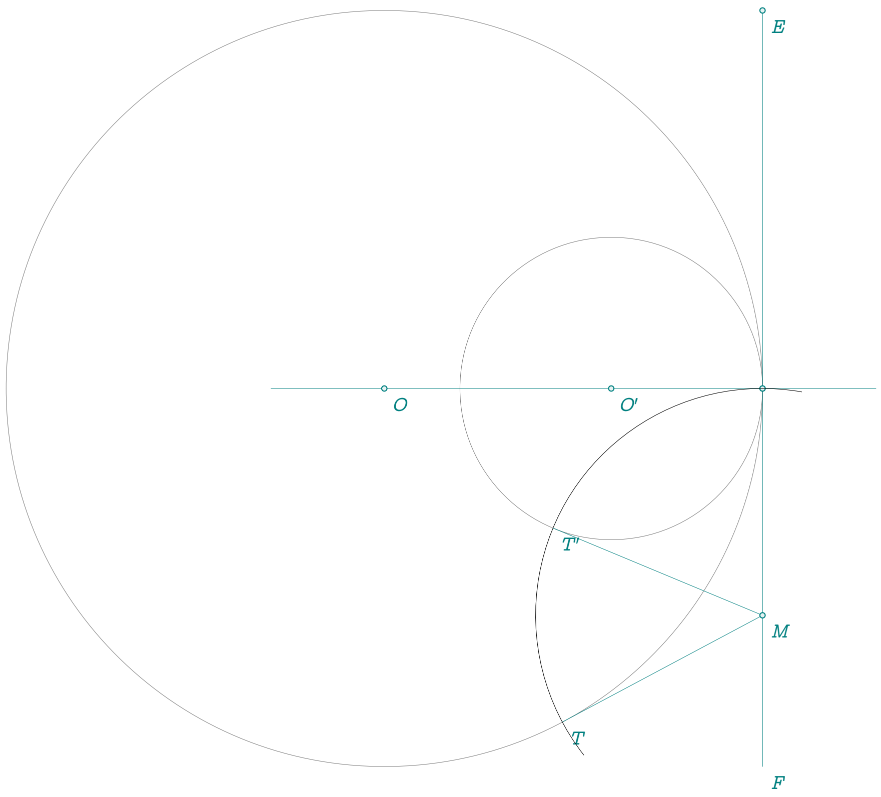
23.28 Radical axis v3

```
   \begin{tkzelements}
   z.O      = point : new (0,0)
   z.B      = point : new (4,0)
   z.Op      = point : new (6,0)
   C.OB      = circle :    new (z.O,z.B)
   C.OpB     = circle :    new (z.Op,z.B)
   L.EF      = C.OB : radical_axis (C.OpB)
   z.E,z.F   = get_points(L.EF)
   z.M       = L.EF : point (0.2)
   L         = C.OB : tangent_from (z.M)
   _,z.T     = get_points (L)
   L         = C.OpB : tangent_from (z.M)
   _,z.Tp    = get_points (L)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O,B O',B)
   \tkzDrawSegments(M,T M,T')
   \tkzDrawSegments(E,F)
   \tkzDrawLine[add=.5 and .5](O,O')
   \tkzDrawPoints(O,B,O',E,F,M,T,T')
   \tkzLabelPoints(O,O',B,E,F,T,T')
   \tkzDrawArc(M,T')(T)
\end{tikzpicture}
```



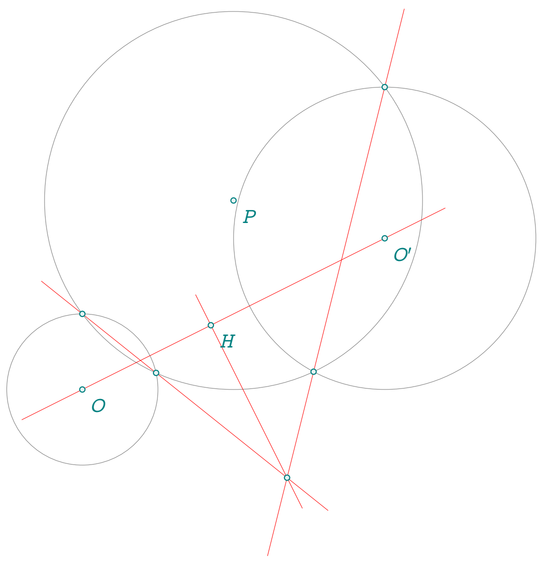
23.29 Radical axis v4

```
\begin{tkzelements}
   z.O     = point : new (0,0)
   z.B     = point : new (5,0)
   z.Op    = point : new (3,0)
   C.OB    = circle :    new (z.O,z.B)
   C.OpB   = circle :    new (z.Op,z.B)
   L.EF    = C.OB : radical_axis (C.OpB)
   z.E,z.F = get_points(L.EF)
   z.H     = L.EF.mid
   z.M     = L.EF : point (.8)
   _,L     = C.OB : tangent_from (z.M)
   _,z.T   = get_points (L)
   _,L     = C.OpB : tangent_from (z.M)
   _,z.Tp  = get_points (L)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O,B O',B)
   \tkzDrawSegments(M,T M,T')
   \tkzDrawSegments(E,F)
   \tkzDrawLine[add=.3 and .3](O,H)
   \tkzDrawPoints(O,O',B,E,H,M)
   \tkzLabelPoints[below right](O,O',E,F,M,T,T')
   \tkzDrawArc(M,B)(T)
\end{tikzpicture}
```



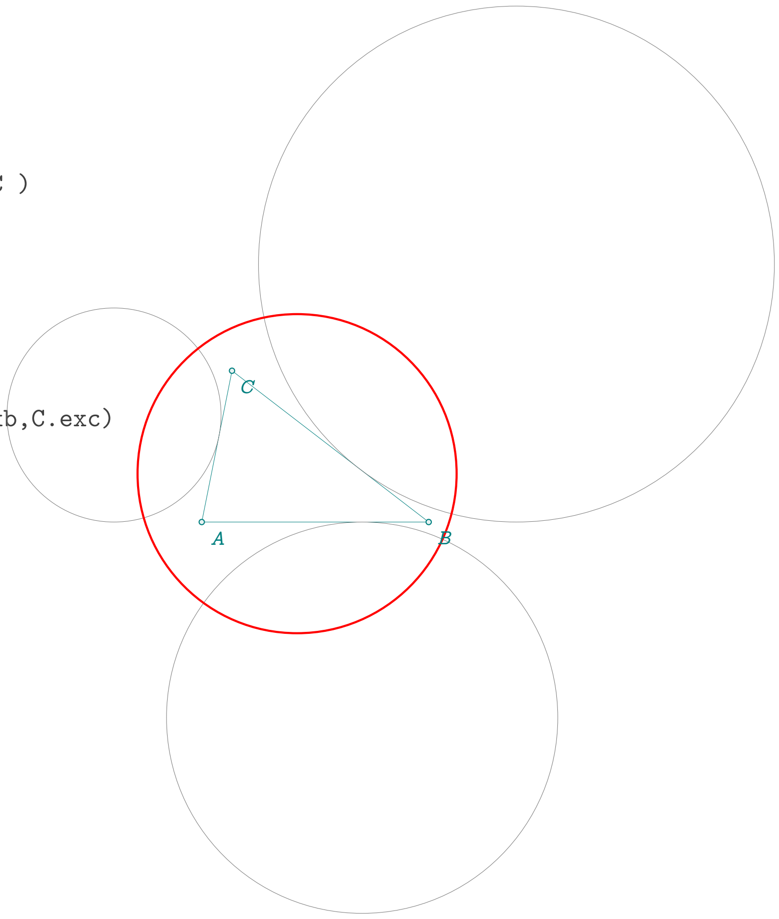
23.30 Radical center

```
\begin{tkzelements}
   z.O      = point : new (0,0)
   z.x      = point : new (1,0)
   z.y      = point : new (4,0)
   z.z      = point : new (2,0)
   z.Op     = point : new (4,2)
   z.P      = point : new (2,2.5)
   C.Ox     = circle :    new (z.O,z.x)
   C.Pz     = circle :    new (z.P,z.z)
   C.Opy    = circle :    new (z.Op,z.y)
   z.ap,z.a = intersection (C.Ox,C.Pz)
   z.bp,z.b = intersection (C.Opy,C.Pz)
   L.aap    = line : new (z.a,z.ap)
   L.bbp    = line : new (z.b,z.bp)
   z.X      = intersection (L.aap,L.bbp)
-- or z.X   = radical_center(C.Ox,C.Pz,C.Opy)
   L.OOp    = line : new (z.O,z.Op)
   z.H      = L.OOp : projection (z.X)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O,a O',b P,z)
   \tkzDrawLines[red](a,X b',X H,X O,O')
   \tkzDrawPoints(O,O',P,a,a',b,b',X,H)
   \tkzLabelPoints[below right](O,O',P,H)
\end{tikzpicture}
```



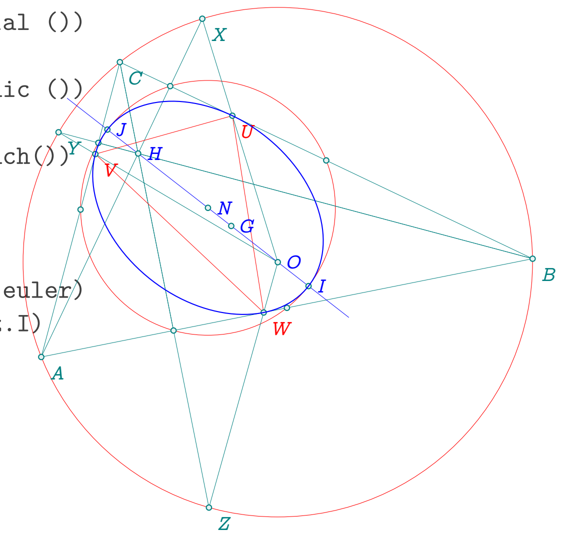
23.31 Radical circle

```
\begin{tkzelements}
   scale        = .25
   z.A          = point: new (0,0)
   z.B          = point: new (6,0)
   z.C          = point: new (0.8,4)
   T.ABC        = triangle : new ( z.A,z.B,z.C )
   C.exa        = T.ABC : ex_circle ()
   z.I_a,z.Xa   = get_points (C.exa)
   C.exb        = T.ABC : ex_circle (1)
   z.I_b,z.Xb   = get_points (C.exb)
   C.exc        = T.ABC : ex_circle (2)
   z.I_c,z.Xc   = get_points (C.exc)
   C.ortho      = C.exa : radical_circle (C.exb,C.exc)
   z.w,z.a      = get_points (C.ortho)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawCircles(I_a,Xa I_b,Xb I_c,Xc)
   \tkzDrawCircles[red,thick](w,a)
   \tkzDrawPoints(A,B,C)
   \tkzLabelPoints(A,B,C)
\end{tikzpicture}
```



23.32 Euler ellipse

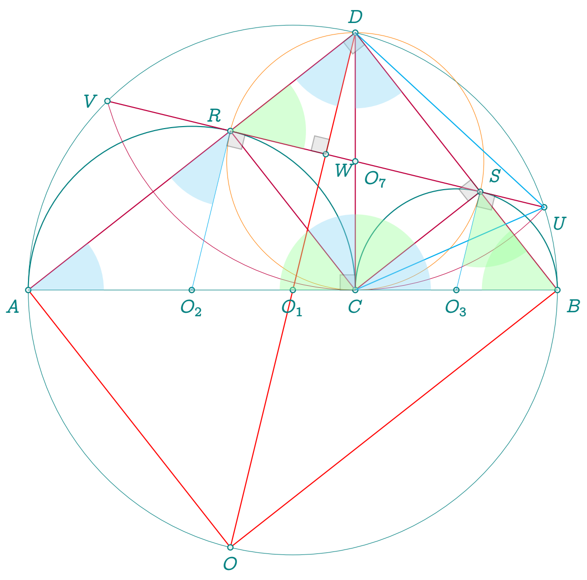
```
\begin{tkzelements}
  scale          = 1.3
  z.A            = point: new (0 , 0)
  z.B            = point: new (5 , 1)
  L.AB           = line : new (z.A,z.B)
  z.C            = point: new (.8 , 3)
  T.ABC          = triangle: new (z.A,z.B,z.C)
  z.N            = T.ABC.eulercenter
  z.G            = T.ABC.centroid
  z.O            = T.ABC.circumcenter
  z.H            = T.ABC.orthocenter
  z.Ma,z.Mb,
  z.Mc           = get_points (T.ABC : medial ())
  z.Ha,z.Hb,
  z.Hc           = get_points (T.ABC : orthic ())
  z.Ea,z.Eb,
  z.Ec           = get_points (T.ABC: extouch())
  L.euler        = T.ABC : euler_line ()
  C.circum       = T.ABC : circum_circle ()
  C.euler        = T.ABC : euler_circle ()
  z.I,z.J        = intersection (L.euler,C.euler)
  E              = ellipse: foci (z.H,z.O,z.I)
  a              = E.Rx
  b              = E.Ry
  ang            = math.deg(E.slope)
  L.AH           = line: new (z.A,z.H)
  L.BH           = line: new (z.B,z.H)
  L.CH           = line: new (z.C,z.H)
  z.X            = intersection (L.AH,C.circum)
  _,z.Y          = intersection (L.BH,C.circum)
  _,z.Z          = intersection (L.CH,C.circum)
  L.BC           = line: new (z.B,z.C)
  L.XO           = line: new (z.X,z.O)
  L.YO           = line: new (z.Y,z.O)
  L.ZO           = line: new (z.Z,z.O)
  z.x            = intersection (L.BC,L.XO)
  z.U            = intersection (L.XO,E)
  _,z.V          = intersection (L.YO,E)
  _,z.W          = intersection (L.ZO,E)
\end{tkzelements}
```



```
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon(A,B,C)
   \tkzDrawCircles[red](N,Ma O,A)
   \tkzDrawSegments(A,X B,Y C,Z B,Hb C,Hc X,O Y,O Z,O)
   \tkzDrawPolygon[red](U,V,W)
   \tkzLabelPoints[red](U,V,W)
   \tkzLabelPoints(A,B,C,X,Y,Z)
   \tkzDrawLine[blue](I,J)
   \tkzLabelPoints[blue,right](O,N,G,H,I,J)
   \tkzDrawPoints(I,J,U,V,W)
   \tkzDrawPoints(A,B,C,N,G,H,O,X,Y,Z,Ma,Mb,Mc,Ha,Hb,Hc)
   \tkzDrawEllipse[blue](N,\tkzUseLua{a},\tkzUseLua{b},\tkzUseLua{ang})
\end{tikzpicture}
```

23.33 Gold Arbelos properties

```
\begin{tkzelements}
  z.A       = point : new(0,0)
  z.B       = point : new(10,0)
  z.C       = gold_segment_ (z.A,z.B)
  L.AB      = line:new (z.A,z.B)
  z.O_1     = L.AB.mid
  L.AC      = line:new (z.A,z.C)
  z.O_2     = L.AC.mid
  L.CB      = line:new (z.C,z.B)
  z.O_3     = L.CB.mid
  C1        = circle:new (z.O_1,z.B)
  C2        = circle:new (z.O_2,z.C)
  C3        = circle:new (z.O_3,z.B)
  z.Q       = C2.north
  z.P       = C3.north
  L1        = line:new (z.O_2,z.O_3)
  z.M_0     = L1:harmonic_ext (z.C)
  L2        = line:new (z.O_1,z.O_2)
  z.M_1     = L2:harmonic_int (z.A)
  L3        = line:new (z.O_1,z.O_3)
  z.M_2     = L3:harmonic_int (z.B)
  Lbq       = line:new (z.B,z.Q)
  Lap       = line:new (z.A,z.P)
  z.S       = intersection (Lbq,Lap)
  z.x       = z.C: north ()
  L         = line : new (z.C,z.x)
  z.D,_     = intersection (L,C1)
  L.CD      = line :new (z.C,z.D)
  z.O_7     = L.CD.mid
  C.DC      = circle: new (z.D,z.C)
  z.U,z.V   = intersection (C.DC,C1)
  L.UV      = line :new (z.U,z.V)
  z.R ,z.S  = L.UV : projection (z.O_2,z.O_3)
  L.O1D     = line : new (z.O_1,z.D)
  z.W       = intersection (L.UV,L.O1D)
  z.O       = C.DC : inversion (z.W)
\end{tkzelements}
```

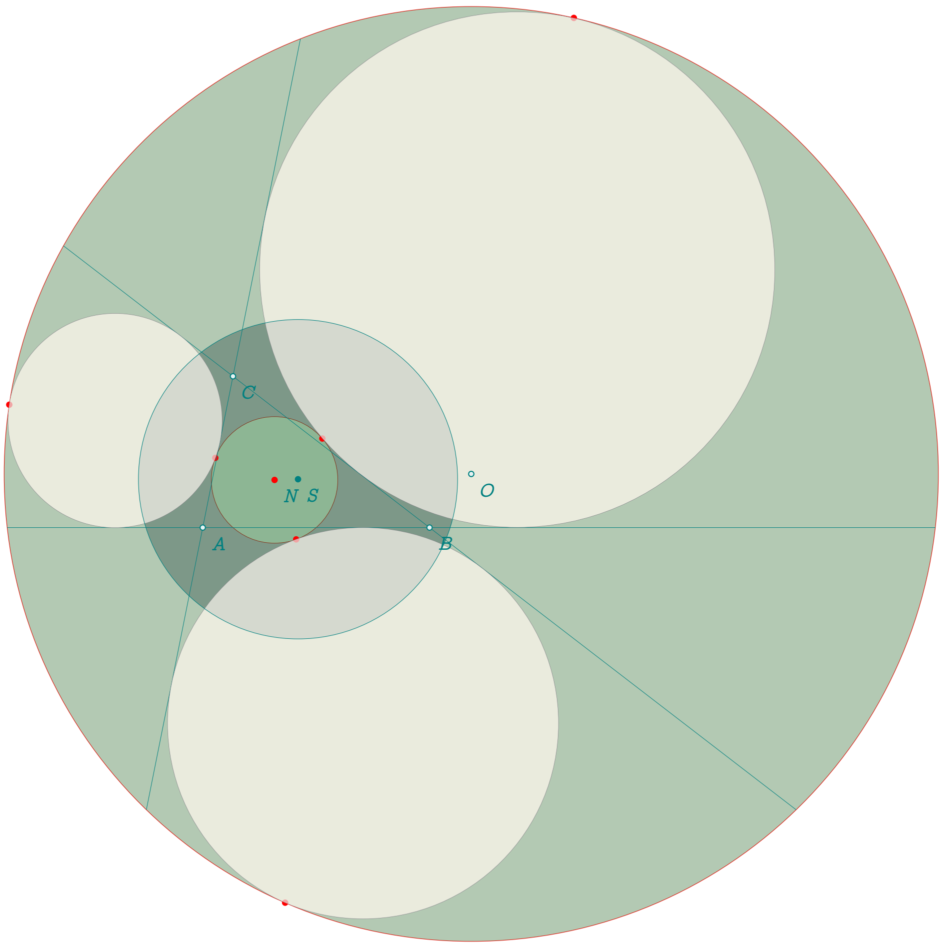


```
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles[teal](O_1,B)
   \tkzDrawSemiCircles[thin,teal](O_2,C O_3,B)
   \tkzDrawArc[purple,delta=0](D,V)(U)
   \tkzDrawCircle[new](O_7,C)
   \tkzDrawSegments[thin,purple](A,D D,B C,R C,S C,D U,V)
   \tkzDrawSegments[thin,red](O,D A,O O,B)
   \tkzDrawPoints(A,B,C,D,O_7) %,
   \tkzDrawPoints(O_1,O_2,O_3,U,V,R,S,W,O)
   \tkzDrawSegments[cyan](O_3,S O_2,R)
   \tkzDrawSegments[very thin](A,B)
   \tkzDrawSegments[cyan,thin](C,U U,D)
   \tkzMarkRightAngles[size=.2,fill=gray!40,opacity=.4](D,C,A A,D,B
     D,S,C D,W,V O_3,S,U O_2,R,U)
   \tkzFillAngles[cyan!40,opacity=.4](B,A,D A,D,O_1
     C,D,B D,C,R B,C,S A,R,O_2)
   \tkzFillAngles[green!40,opacity=.4](S,C,D W,R,D
     D,B,C R,C,A O_3,S,B)
   \tkzLabelPoints[below](C,O_2,O_3,O_1)
   \tkzLabelPoints[above](D)
   \tkzLabelPoints[below](O)
   \tkzLabelPoints[below left](A)
   \tkzLabelPoints[above left](R)
   \tkzLabelPoints[above right](S)
   \tkzLabelPoints[left](V)
   \tkzLabelPoints[below right](B,U,W,O_7)
\end{tikzpicture}
```

23.34 Apollonius circle v1 with inversion

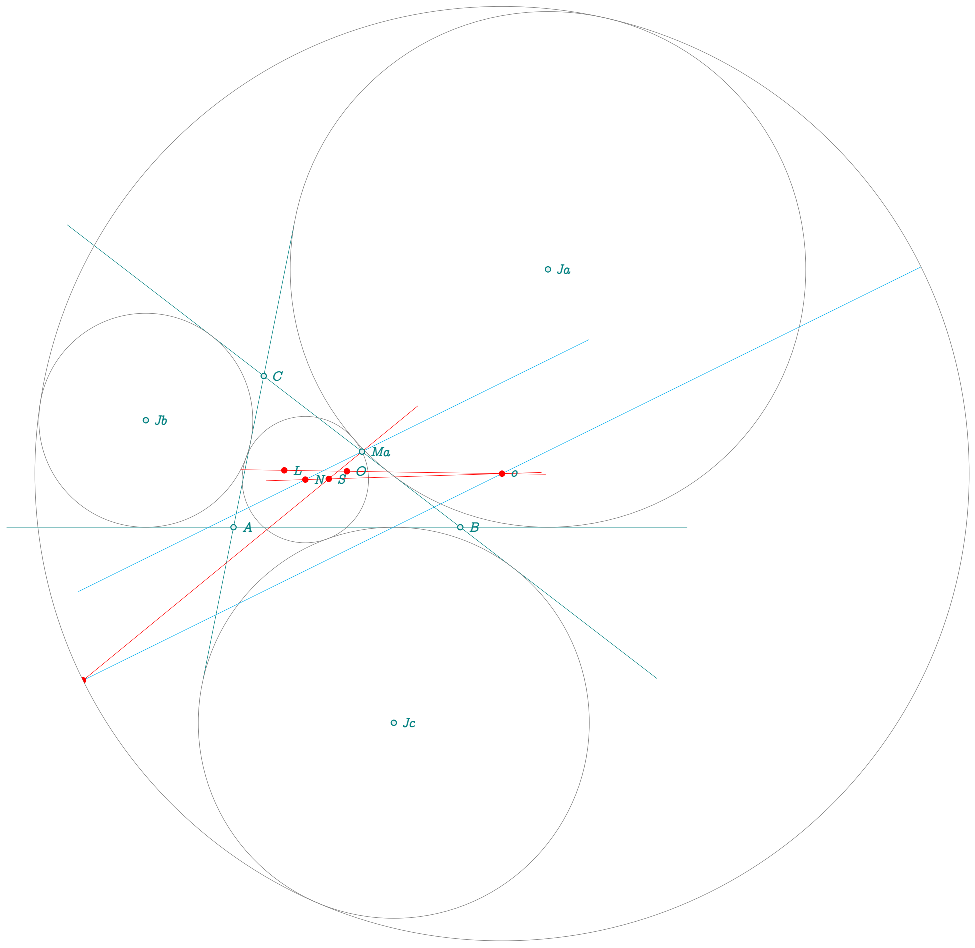
```
\begin{tkzelements}
   scale             = .7
   z.A               = point: new (0,0)
   z.B               = point: new (6,0)
   z.C               = point: new (0.8,4)
   T.ABC             = triangle : new ( z.A,z.B,z.C )
   z.N               = T.ABC.eulercenter
   z.Ea,z.Eb,z.Ec    = get_points ( T.ABC : feuerbach () )
   z.Ja,z.Jb,z.Jc    = get_points ( T.ABC : excentral () )
   z.S               = T.ABC : spieker_center ()
   C.JaEa            = circle : new (z.Ja,z.Ea)
   C.ortho           = circle : radius (z.S,math.sqrt(C.JaEa : power (z.S) ))
   z.a               = C.ortho.south
   C.euler           = T.ABC: euler_circle ()
   C.apo             = C.ortho : inversion (C.euler)
   z.O               = C.apo.center
   z.xa,z.xb,z.xc    = C.ortho : inversion (z.Ea,z.Eb,z.Ec)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
\tkzDrawCircles[red](O,xa N,Ea)
\tkzFillCircles[green!30!black,opacity=.3](O,xa)
\tkzFillCircles[yellow!30,opacity=.7](Ja,Ea Jb,Eb Jc,Ec)
\tkzFillCircles[teal!30!black,opacity=.3](S,a)
\tkzFillCircles[green!30,opacity=.3](N,Ea)
\tkzDrawPoints[red](Ea,Eb,Ec,xa,xb,xc,N)
\tkzClipCircle(O,xa)
\tkzDrawLines[add=3 and 3](A,B A,C B,C)
\tkzDrawCircles(Ja,Ea Jb,Eb Jc,Ec)
\tkzFillCircles[lightgray!30,opacity=.7](Ja,Ea Jb,Eb Jc,Ec)
\tkzDrawCircles[teal](S,a)
\tkzDrawPoints(A,B,C,O)
```

```
\tkzDrawPoints[teal](S)
\tkzLabelPoints(A,B,C,O,S,N)
\end{tikzpicture}
```



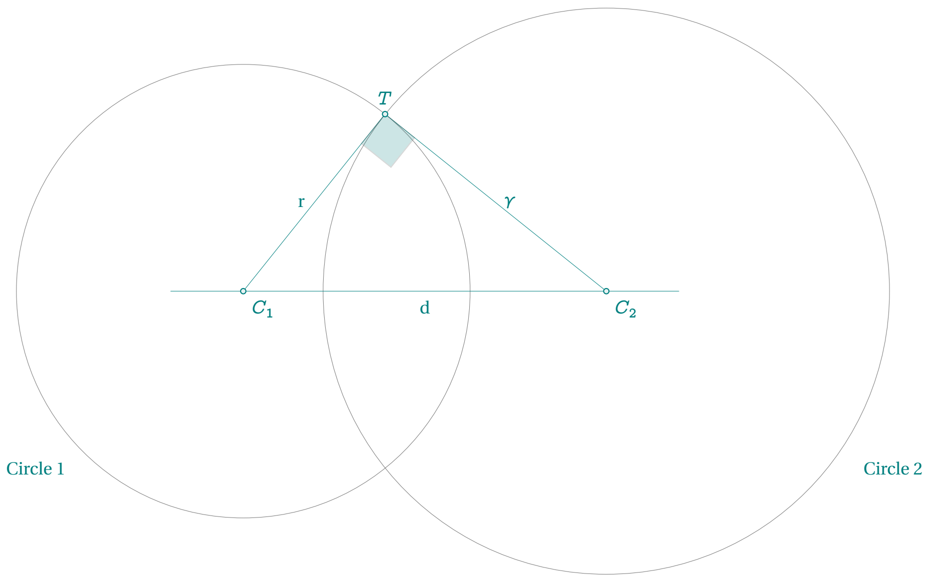
23.35 Apollonius circle v2

```
\begin{tkzelements}
   scale        = .5
   z.A          = point: new (0,0)
   z.B          = point: new (6,0)
   z.C          = point: new (0.8,4)
   T.ABC        = triangle: new(z.A,z.B,z.C)
   z.O          = T.ABC.circumcenter
   z.H          = T.ABC.orthocenter
   z.G          = T.ABC.centroid
   z.L          = T.ABC: lemoine_point ()
   z.S          = T.ABC: spieker_center ()
   C.euler      = T.ABC: euler_circle ()
   z.N,z.Ma     = get_points (C.euler)
   C.exA        = T.ABC : ex_circle ()
   z.Ja,z.Xa    = get_points (C.exA)
   C.exB        = T.ABC : ex_circle (1)
   z.Jb,z.Xb    = get_points (C.exB)
   C.exC        = T.ABC : ex_circle (2)
   z.Jc,z.Xc    = get_points (C.exC)
   L.OL         = line: new (z.O,z.L)
   L.NS         = line: new (z.N,z.S)
   z.o          = intersection (L.OL,L.NS) -- center of Apollonius circle
   L.NMa        = line: new (z.N,z.Ma)
   L.ox         = L.NMa: ll_from (z.o)
   L.MaS        = line: new (z.Ma,z.S)
   z.t          = intersection (L.ox,L.MaS) -- through
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawLines[add=1 and 1](A,B A,C B,C)
   \tkzDrawCircles(Ja,Xa Jb,Xb Jc,Xc o,t N,Ma) %
   \tkzClipCircle(o,t)
   \tkzDrawLines[red](o,L N,o Ma,t)
   \tkzDrawLines[cyan,add=4 and 4](Ma,N o,t)
   \tkzDrawPoints(A,B,C,Ma,Ja,Jb,Jc)
   \tkzDrawPoints[red](N,O,L,S,o,t)
   \tkzLabelPoints[right,font=\tiny](A,B,C,Ja,Jb,Jc,O,N,L,S,Ma,o)
\end{tikzpicture}
```



23.36 Orthogonal circles v1

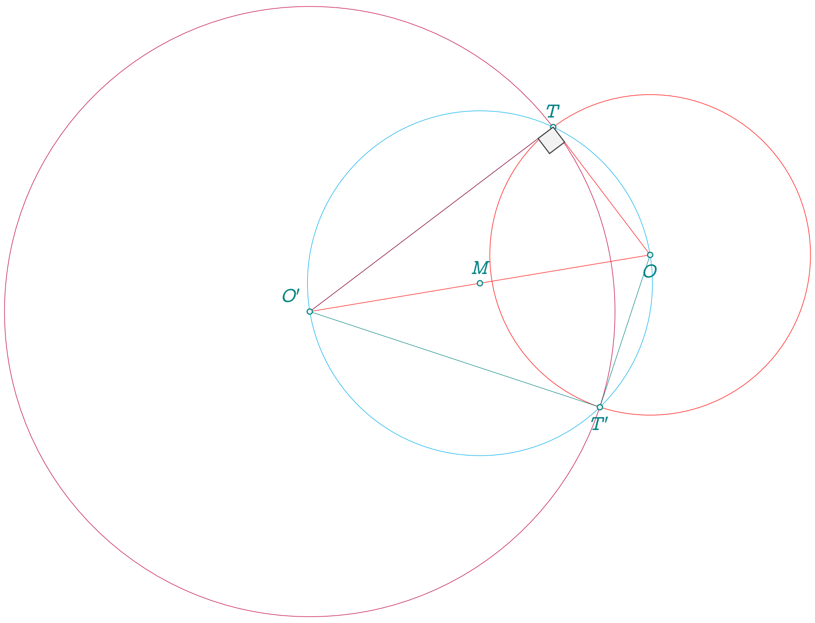
```
\begin{tkzelements}
   scale    = .6
   z.C_1    = point: new (0,0)
   z.C_2    = point: new (8,0)
   z.A      = point: new (5,0)
   C        = circle: new (z.C_1,z.A)
   z.S,z.T  = get_points (C: orthogonal_from (z.C_2))
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(C_1,T C_2,T)
   \tkzDrawSegments(C_1,T C_2,T)
   \tkzDrawLine(C_1,C_2)
   \tkzMarkRightAngle[fill=teal,%
  opacity=.2,size=1](C_1,T,C_2)
   \tkzDrawPoints(C_1,C_2,T)
   \tkzLabelPoints(C_1,C_2)
   \tkzLabelPoints[above](T)
   \tkzLabelSegment[left](C_1,T){r}
   \tkzLabelSegments[right](C_2,T){$\gamma$}
   \tkzLabelSegment[below](C_1,C_2){d}
   \tkzLabelCircle[left=10pt](C_1,T)(180){Circle 1}
   \tkzLabelCircle[right=10pt](C_2,T)(180){Circle 2}
\end{tikzpicture}
```



23.37 Orthogonal circles v2

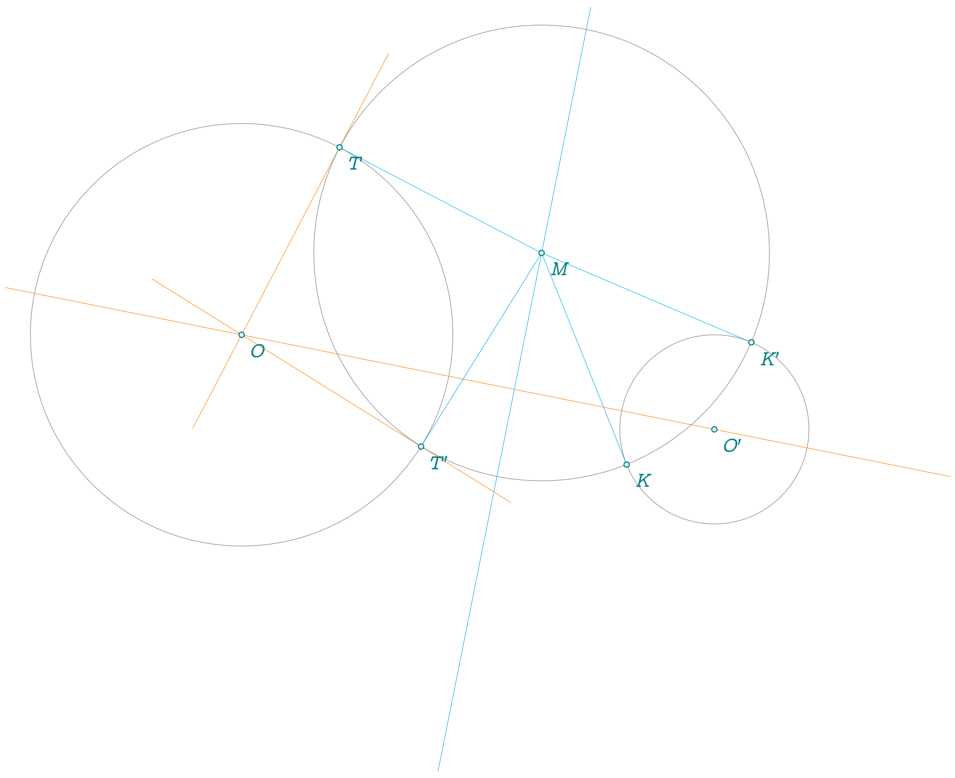
```
\begin{tkzelements}
scale     = .75
z.O       = point: new (2,2)
z.Op      = point: new (-4,1)
z.P       = point: polar (4,0)
C.OP      = circle: new (z.O,z.P)
C.Oz1     =  C.OP : orthogonal_from (z.Op)
z.z1      = C.Oz1.through
L.OP      = line : new (z.O,z.P)
C.Opz1    = circle: new (z.Op,z.z1)
L.T,L.Tp = C.Opz1 : tangent_from (z.O)
```

```
z.T      = L.T.pb
z.Tp     = L.Tp.pb
L.OOp    = line : new (z.O,z.Op)
z.M      = L.OOp.mid
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle[red](O,P)
   \tkzDrawCircle[purple](O',z1)
   \tkzDrawCircle[cyan](M,T)
   \tkzDrawSegments(O',T O,T' O',T')
   \tkzDrawSegment[purple](O',T)
   \tkzDrawSegments[red](O,T O,O')
   \tkzDrawPoints(O,O',T,T',M)
   \tkzMarkRightAngle[fill=gray!10](O',T,O)
   \tkzLabelPoint[below](O){$O$}
   \tkzLabelPoint[above](T){$T$}
   \tkzLabelPoint[above](M){$M$}
   \tkzLabelPoint[below](T'){$T'$}
   \tkzLabelPoint[above left](O'){$O'$}
\end{tikzpicture}
```

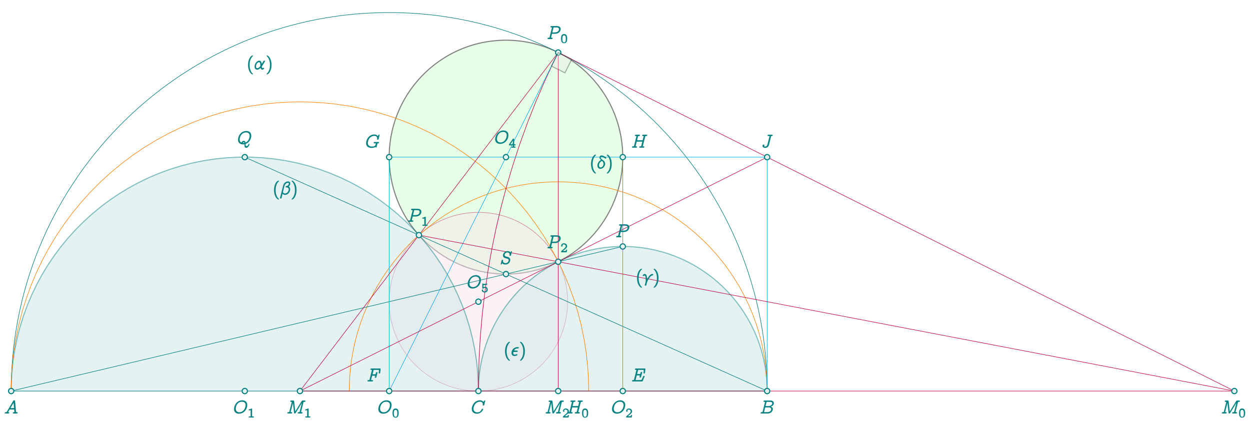


23.38 Orthogonal circle to two circles

```
\begin{tkzelements}
   z.O         = point :   new (-1,0)
   z.B         = point :   new (0,2)
   z.Op        = point :   new (4,-1)
   z.D         = point :   new (4,0)
   C.OB        = circle :  new (z.O,z.B)
   C.OpD       = circle :  new (z.Op,z.D)
   z.E,z.F     = get_points (C.OB : radical_axis (C.OpD))
   L.EF        = line : new (z.E,z.F)
   z.M         = L.EF : point (.25)
   L.T,L.Tp     = C.OB : tangent_from (z.M)
   L.K,L.Kp     = C.OpD : tangent_from (z.M)
   z.T          = L.T.pb
   z.K          = L.K.pb
   z.Tp         = L.Tp.pb
   z.Kp         = L.Kp.pb
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(O,B O',D)
   \tkzDrawLine[cyan](E,F)
   \tkzDrawLines[add=.5 and .5,orange](O,O' O,T O,T')
   \tkzDrawSegments[cyan](M,T M,T' M,K M,K')
   \tkzDrawCircle(M,T)
   \tkzDrawPoints(O,O',T,M,T',K,K')
   \tkzLabelPoints(O,O',T,T',M,K,K')
\end{tikzpicture}
```



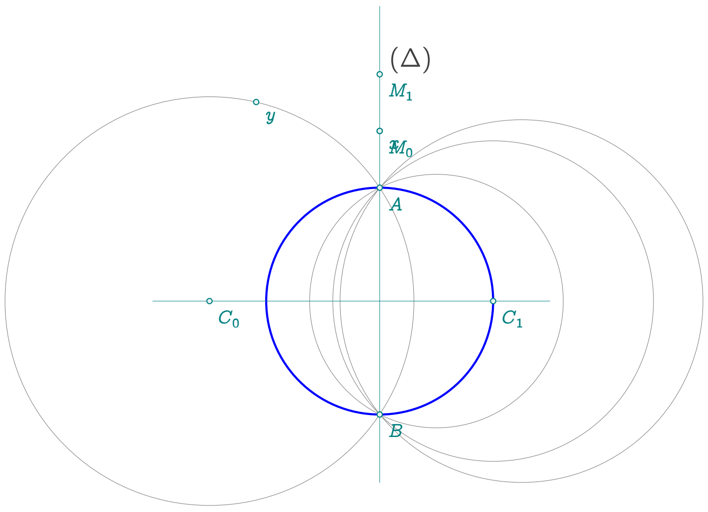
23.39 Midcircles



```
\begin{tkzelements}
   z.A       = point: new (0 , 0)
   z.B       = point: new (10 , 0)
   L.AB      = line : new (z.A,z.B)
   z.C       = L.AB: gold_ratio ()
   L.AC      = line : new (z.A,z.C)
   L.CB      = line : new (z.C,z.B)
   z.O_0     = L.AB.mid
   z.O_1     = L.AC.mid
   z.O_2     = L.CB.mid
   C.O0B     = circle : new (z.O_0,z.B)
   C.O1C     = circle : new (z.O_1,z.C)
   C.O2C     = circle : new (z.O_2,z.B)
   z.Q       = C.O1C : midarc (z.C,z.A)
   z.P       = C.O2C : midarc (z.B,z.C)
   L.O1O2    = line : new (z.O_1,z.O_2)
   L.O0O1    = line : new (z.O_0,z.O_1)
   L.O0O2    = line : new (z.O_0,z.O_2)
   z.M_0     = L.O1O2 : harmonic_ext (z.C)
   z.M_1     = L.O0O1 : harmonic_int (z.A)
   z.M_2     = L.O0O2 : harmonic_int (z.B)
   L.BQ      = line : new (z.B,z.Q)
   L.AP      = line : new (z.A,z.P)
   z.S       = intersection (L.BQ,L.AP)
   L.CS      = line : new (z.C,z.S)
   C.M1A     = circle : new (z.M_1,z.A)
   C.M2B     = circle : new (z.M_2,z.B)
   z.P_0     = intersection (L.CS,C.O0B)
   z.P_1     = intersection (C.M2B,C.O1C)
   z.P_2     = intersection (C.M1A,C.O2C)
   T.P012    = triangle : new (z.P_0,z.P_1,z.P_2)
   z.O_4     = T.P012.circumcenter
   T.CP12    = triangle : new (z.C,z.P_1,z.P_2)
   z.O_5     = T.CP12.circumcenter
   z.BN      = z.B : north ()
   L.BBN     = line : new (z.B,z.BN)
   L.M1P2    = line : new (z.M_1,z.P_2)
   z.J      = intersection (L.BBN,L.M1P2)
   L.AP0    = line : new (z.A,z.P_0)
   L.BP0    = line : new (z.B,z.P_0)
   C.O4P0   = circle : new (z.O_4,z.P_0)
   _,z.G    = intersection (L.AP0,C.O4P0)
   z.H      = intersection (L.BP0,C.O4P0)
   z.Ap     = z.M_1: symmetry (z.A)
   z.H_4,z.F,z.E,z.H_0 = L.AB : projection (z.O_4,z.G,z.H,z.P_0)
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawCircle[thin,fill=green!10](O_4,P_0)
\tkzDrawCircle[purple,fill=purple!10,opacity=.5](O_5,C)
\tkzDrawSemiCircles[teal](O_0,B)
\tkzDrawSemiCircles[thin,teal,fill=teal!20,opacity=.5](O_1,C O_2,B)
\tkzDrawSemiCircles[color = orange](M_2,B)
\tkzDrawSemiCircles[color = orange](M_1,A')
\tkzDrawArc[purple,delta=0](M_0,P_0)(C)
\tkzDrawSegments[very thin](A,B A,P B,Q)
\tkzDrawSegments[color=cyan](O_0,P_0 B,J G,J G,O_0 H,O_2)
\tkzDrawSegments[ultra thin,purple](M_1,P_0 M_2,P_0 M_1,M_0 M_0,P_1 M_0,P_0 M_1,J)
\tkzDrawPoints(A,B,C,P_0,P_2,P_1,M_0,M_1,M_2,J,P,Q,S)
\tkzDrawPoints(O_0,O_1,O_2,O_4,O_5,G,H)
\tkzMarkRightAngle[size=.2,fill=gray!20,opacity=.4](O_0,P_0,M_0)
\tkzLabelPoints[below](A,B,C,M_0,M_1,M_2,O_1,O_2,O_0)
\tkzLabelPoints[above](P_0,O_5,O_4)
\tkzLabelPoints[above](P_1,J)
\tkzLabelPoints[above](P_2,P,Q,S)
\tkzLabelPoints[above right](H,E)
\tkzLabelPoints[above left](F,G)
\tkzLabelPoints[below right](H_0)
\tkzLabelCircle[below=4pt,font=\scriptsize](O_1,C)(80){$(\beta)$}
\tkzLabelCircle[below=4pt,font=\scriptsize](O_2,B)(80){$(\gamma)$}
\tkzLabelCircle[below=4pt,font=\scriptsize](O_0,B)(110){$(\alpha)$}
\tkzLabelCircle[left,font=\scriptsize](O_4,P_2)(60){$(\delta)$}
\tkzLabelCircle[above left,font=\scriptsize](O_5,C)(40){$(\epsilon)$}
\end{tikzpicture}
```

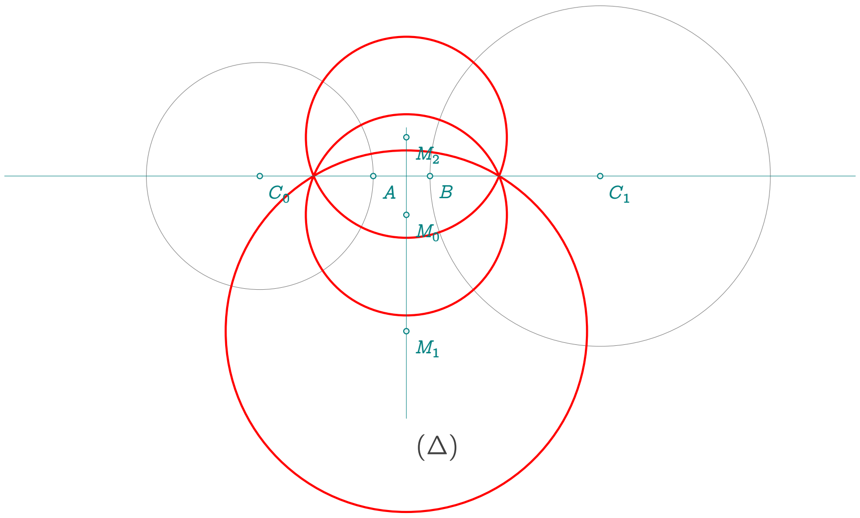
23.40 Pencil v1

```
\begin{tkzelements}
   scale       = .75
   z.A         = point : new (0,2)
   z.B         = point : new (0,-2)
   z.C_0       = point : new (-3,0)
   z.C_1       = point : new (2,0)
   z.C_3       = point : new (2.5,0)
   z.C_5       = point : new (1,0)
   L.BA        = line : new (z.B,z.A)
   z.M_0       = L.BA : point (1.25)
   z.M_1       = L.BA : point (1.5)
   C.C0A       = circle :    new (z.C_0,z.A)
   z.x,z.y     = get_points (C.C0A : orthogonal_from (z.M_0))
   z.xp,z.yp   = get_points (C.C0A : orthogonal_from (z.M_1))
   z.O         = L.BA.mid
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(C_0,A C_1,A C_3,A C_5,A)
   \tkzDrawCircles[thick,color=red](M_0,x M_1,x')
   \tkzDrawCircles[thick,color=blue](O,A)
   \tkzDrawLines(C_0,C_1 B,M_1)
   \tkzDrawPoints(A,B,C_0,C_1,M_0,M_1,x,y)
   \tkzLabelPoints[below right](A,B,C_0,C_1,M_0,M_1,x,y)
   \tkzLabelLine[pos=1.25,right]( M_0,M_1){$(\Delta)$}
\end{tikzpicture}
```



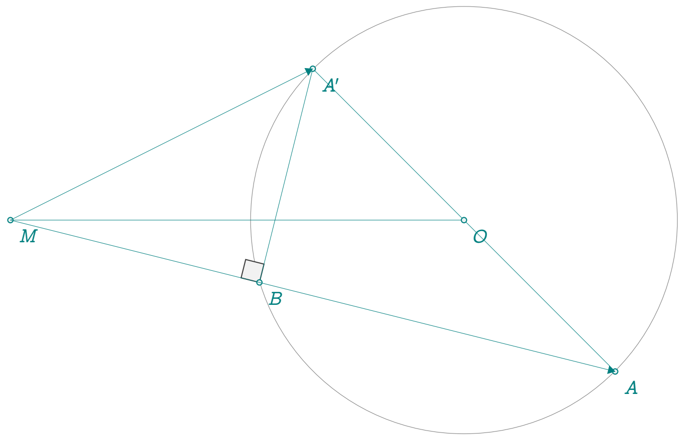
23.41 Pencil v2

```
\begin{tkzelements}
   scale=.75
   z.A     = point : new (0,0)
   z.B     = point : new (1,0)
   z.C_0   = point : new (-2,0)
   z.C_1   = point : new (4,0)
   C.C0A   = circle : new (z.C_0,z.A)
   C.C1B   = circle : new (z.C_1,z.B)
   L.EF    = C.C0A : radical_axis (C.C1B)
   z.M_0   = L.EF : point (.4)
   z.M_1   = L.EF : point (.1)
   z.M_2   = L.EF : point (.6)
   C.orth0     = C.C0A : orthogonal_from (z.M_0)
   C.orth1     = C.C0A : orthogonal_from (z.M_1)
   C.orth2     = C.C0A : orthogonal_from (z.M_2)
   z.u         = C.orth0.through
   z.v         = C.orth1.through
   z.t         = C.orth2.through
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(C_0,A C_1,B)
   \tkzDrawCircles[thick,color=red](M_0,u M_1,v M_2,t)
   \tkzDrawLines[add= .75 and .75](C_0,C_1 M_0,M_1)
   \tkzDrawPoints(A,B,C_0,C_1,M_0,M_1,M_2)
   \tkzLabelPoints[below right](A,B,C_0,C_1,M_0,M_1,M_2)
   \tkzLabelLine[pos=2,right]( M_0,M_1){$(\Delta)$}
\end{tikzpicture}
```

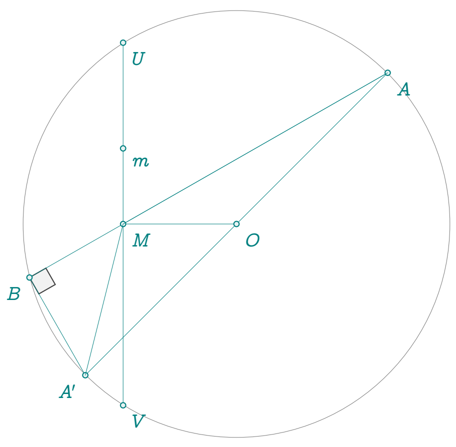


23.42 Power v1

```
\begin{tkzelements}
   z.O      = point : new (0,0)
   z.A      = point : new (2,-2)
   z.M      = point : new (-6,0)
   L.AM     = line : new (z.A,z.M)
   C.OA     = circle :    new (z.O,z.A)
   z.Ap     = C.OA : antipode (z.A)
   z.B      = intersection (L.AM, C.OA)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(O,A)
   \tkzMarkRightAngle[fill=gray!10](A',B,M)
   \tkzDrawSegments(M,O A,A' A',B)
   \tkzDrawPoints(O,A,A',M,B)
   \tkzLabelPoints(O,A,A',M,B)
   \tkzDrawSegments[-Triangle](M,A M,A')
\end{tikzpicture}
```



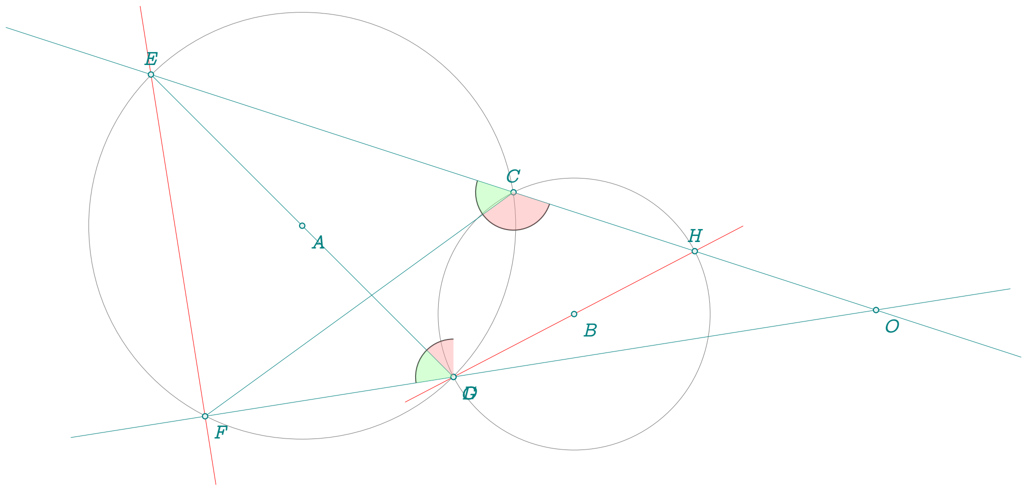
23.43 Power v2



```
\begin{tkzelements}
   z.O      = point : new (0,0)
   z.A      = point : new (2,2)
   z.M      = point : new (-1.5,0)
   L.AM     = line : new (z.A,z.M)
   C.OA     = circle :    new (z.O,z.A)
   z.Ap     = C.OA : antipode (z.A)
   _,z.B    = intersection (L.AM, C.OA)
   z.m      = z.M : north(1)
   L.mM     = line : new (z.m,z.M)
   z.U,z.V = intersection (L.mM,C.OA)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(O,A)
   \tkzMarkRightAngle[fill=gray!10](A',B,M)
   \tkzDrawSegments(M,O A,A' A',B A,B U,V)
   \tkzDrawPoints(O,A,A',M,B,U,V,m)
   \tkzLabelPoints(O,A,M,U,V,m)
   \tkzLabelPoints[below left](A',B)
   \tkzDrawSegments(M,A M,A')
\end{tikzpicture}
```

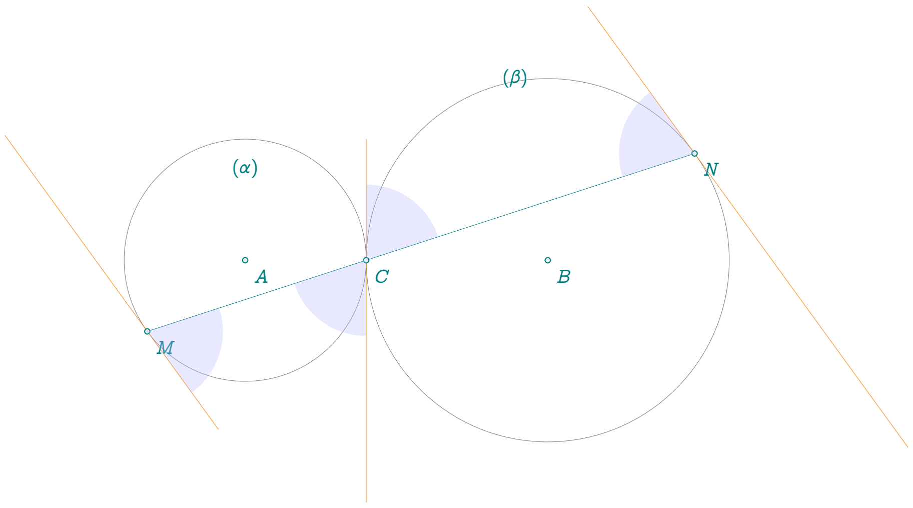
23.44 Reim v1

```
\begin{tkzelements}
   z.A      = point: new (0,0)
   z.E      = point: new (-2,2)
   C.AE     = circle :   new (z.A,z.E)
   z.C      = C.AE : point (0.65)
   z.D      = C.AE : point (0.5)
   z.F      = C.AE : point (0.30)
   L.EC     = line: new (z.E,z.C)
   z.H      = L.EC : point (1.5)
   T.CDH    = triangle : new (z.C,z.D,z.H)
   z.B      = T.CDH.circumcenter
   C.BD     = circle : new (z.B,z.D)
   L.FD     = line: new (z.F,z.D)
   z.G      = intersection (L.FD,C.BD)
   z.O      = intersection (L.EC,L.FD)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,E B,H)
   \tkzDrawSegments(E,D C,F)
   \tkzDrawLines(E,O F,O)
   \tkzDrawLines[red](E,F H,G)
   \tkzDrawPoints(A,...,H,O)
   \tkzLabelPoints(A,B,D,F,G,O)
   \tkzLabelPoints[above](E,C,H)
   \tkzMarkAngles[size=.5](E,C,F E,D,F)
   \tkzFillAngles[green!40,opacity=.4,size=.5](E,C,F E,D,F)
   \tkzMarkAngles[size=.5](F,C,H G,D,E)
   \tkzFillAngles[red!40,opacity=.4,size=.5](F,C,H G,D,E)
\end{tikzpicture}
```



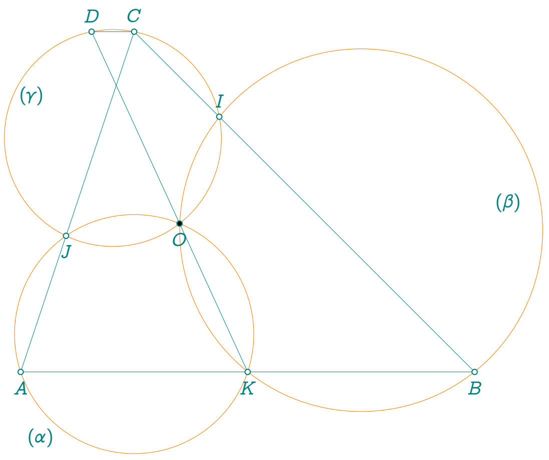
23.45 Reim v2

```
\begin{tkzelements}
   scale     = .6
   z.A       = point: new (0,0)
   z.B       = point: new (10,0)
   z.C       = point: new (4,0)
   C.AC      = circle: new (z.A,z.C)
   z.c,z.cp = get_points (C.AC: tangent_at (z.C))
   z.M       = C.AC: point (0.6)
   L.MC      = line: new (z.M,z.C)
   C.BC      = circle: new (z.B,z.C)
   z.N       = intersection (L.MC,C.BC)
   z.m,z.mp = get_points (C.AC: tangent_at (z.M))
   z.n,z.np = get_points (C.BC: tangent_at (z.N))
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,C B,C)
   \tkzDrawLines[new,add=1 and 1](M,m N,n C,c)
   \tkzDrawSegment(M,N)
   \tkzDrawPoints(A,B,C,M,N)
   \tkzLabelPoints[below right](A,B,C,M,N)
   \tkzFillAngles[blue!30,opacity=.3](m',M,C N,C,c' M,C,c n',N,C)
   \tkzLabelCircle[below=4pt,font=\scriptsize](A,C)(90){$(\alpha)$}
   \tkzLabelCircle[left=4pt,font=\scriptsize](B,C)(-90){$(\beta)$}
\end{tikzpicture}
```



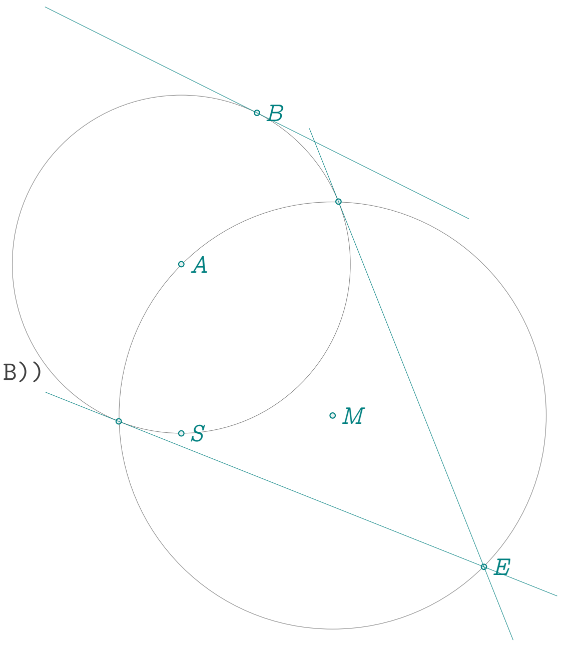
23.46 Reim v3

```
\begin{tkzelements}
   z.A      = point: new (0,0)
   z.B      = point: new (8,0)
   z.C      = point: new (2,6)
   L.AB     = line : new (z.A,z.B)
   L.AC     = line : new (z.A,z.C)
   L.BC     = line : new (z.B,z.C)
   z.I      = L.BC : point (0.75)
   z.J      = L.AC : point (0.4)
   z.K      = L.AB : point (0.5)
   T.AKJ    = triangle : new (z.A,z.K,z.J)
   T.BIK    = triangle : new (z.B,z.I,z.K)
   T.CIJ    = triangle : new (z.C,z.I,z.J)
   z.x      = T.AKJ.circumcenter
   z.y      = T.BIK.circumcenter
   z.z      = T.CIJ.circumcenter
   C.xK     = circle: new (z.x,z.K)
   C.yK     = circle: new (z.y,z.K)
   z.O,_    = intersection (C.xK,C.yK)
   C.zO     = circle: new (z.z,z.O)
   L.KO     = line: new (z.K,z.O)
   z.D      = intersection (L.KO,C.zO)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawSegments(K,D D,C)
   \tkzDrawPolygon[teal](A,B,C)
   \tkzDrawCircles[orange](x,A y,B z,C)
   \tkzDrawPoints[fill=white](A,B,C,I,J,K,D)
   \tkzLabelPoints[below](A,B,J,K,O)
   \tkzLabelPoints[above](C,D,I)
   \tkzDrawPoints[fill=black](O)
   \tkzLabelCircle[below=4pt,font=\scriptsize](x,A)(20){$(\alpha)$}
   \tkzLabelCircle[left=4pt,font=\scriptsize](y,B)(60){$(\beta)$}
   \tkzLabelCircle[below=4pt,font=\scriptsize](z,C)(60){$(\gamma)$}
\end{tikzpicture}
```



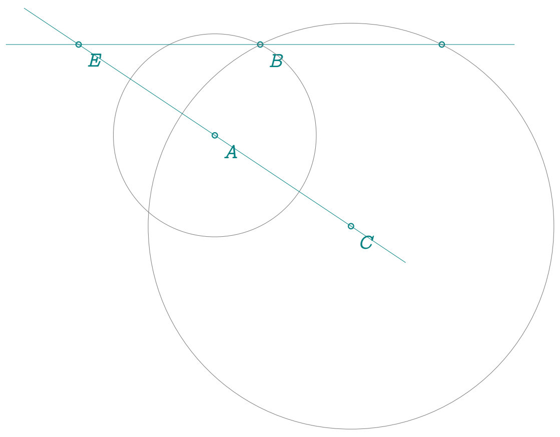
23.47 Tangent and circle

```
\begin{tkzelements}
   z.A          = point:    new (1,0)
   z.B          = point:    new (2,2)
   z.E          = point:    new (5,-4)
   L.AE         = line :    new (z.A,z.E)
   C.AB         = circle:   new (z.A , z.B)
   z.S          = C.AB.south
   z.M          = L.AE.mid
   L.Ti,L.Tj    = C.AB:   tangent_from (z.E)
   z.i          = L.Ti.pb
   z.j          = L.Tj.pb
   z.k,z.l      = get_points (C.AB:  tangent_at (z.B))
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(A,B M,A)
   \tkzDrawPoints(A,B,E,i,j,M,S)
   \tkzDrawLines(E,i E,j k,l)
   \tkzLabelPoints[right,font=\small](A,B,E,S,M)
\end{tikzpicture}
```



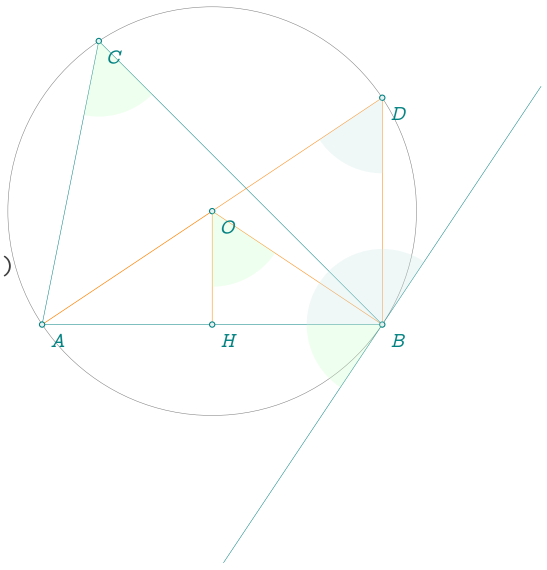
23.48 Homothety

```
\begin{tkzelements}
   z.A      = point:  new (0,0)
   z.B      = point:  new (1,2)
   z.E      = point:  new (-3,2)
   z.C,z.D  = z.E : homothety(2,z.A,z.B)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPoints(A,B,C,E,D)
   \tkzLabelPoints(A,B,C,E)
   \tkzDrawCircles(A,B C,D)
   \tkzDrawLines(E,C E,D)
\end{tikzpicture}
```



23.49 Tangent and chord

```
\begin{tkzelements}
   scale        = .8
   z.A          = point: new (0 , 0)
   z.B          = point: new (6 , 0)
   z.C          = point: new (1 , 5)
   z.Bp         = point: new (2 , 0)
   T.ABC        = triangle: new (z.A,z.B,z.C)
   L.AB         = line: new (z.A,z.B)
   z.O          = T.ABC.circumcenter
   C.OA         = circle: new (z.O,z.A)
   z.D          = C.OA: point (4.5)
   L.AO         = line: new (z.A,z.O)
   z.b1,z.b2    = get_points (C.OA: tangent_at (z.B))
   z.H          = L.AB: projection (z.O)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircle(O,A)
   \tkzDrawPolygon(A,B,C)
   \tkzDrawSegments[new](A,O B,O O,H A,D D,B)
   \tkzDrawLine(b1,b2)
   \tkzDrawPoints(A,B,C,D,H,O)
   \tkzFillAngles[green!20,opacity=.3](H,O,B A,C,B  A,B,b1)
   \tkzFillAngles[teal!20,opacity=.3](A,D,B b2,B,A)
   \tkzLabelPoints(A,B,C,D,H,O)
\end{tikzpicture}
```

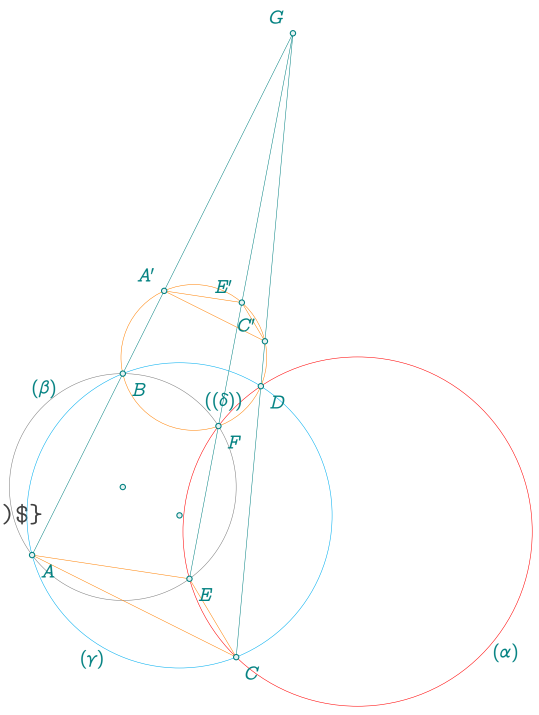


23.50 Three chords

```
\begin{tkzelements}
z.O  = point: new (0 , 0)
z.B  = point: new (0 , 2)
z.P  = point: new (1 , -.5)
C.OB = circle : new (z.O,z.B)
C.PB = circle : new (z.P,z.B)
_,z.A    = intersection (C.OB,C.PB)
z.D  = C.PB: point(0.85)
```

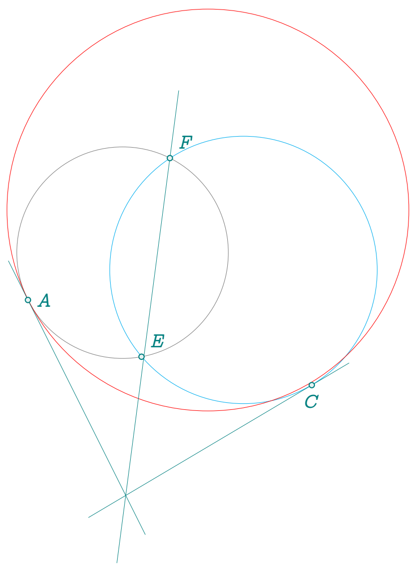
```
z.C  = C.PB: point(0.5)
z.E  = C.OB: point(0.6)
L.AB = line : new (z.A,z.B)
L.CD = line : new (z.C,z.D)
z.G  = intersection (L.AB,L.CD)
L.GE = line : new (z.G,z.E)
z.F,_    = intersection (L.GE,C.OB)
T.CDE    = triangle: new (z.C,z.D,z.E)
T.BFD    = triangle: new (z.B,z.F,z.D)
z.w  = T.CDE.circumcenter
z.x  = T.BFD.circumcenter
L.GB = line : new (z.G,z.B)
L.GE = line : new (z.G,z.E)
L.GD = line : new (z.G,z.D)
C.xB = circle : new (z.x,z.B)
C.xF = circle : new (z.x,z.F)
C.xD = circle : new (z.x,z.D)
z.Ap = intersection (L.GB,C.xB)
z.Ep,_   = intersection (L.GE,C.xF)
z.Cp,_   = intersection (L.GD,C.xD)
\end{tkzelements}
```

```
  \begin{tikzpicture}
     \tkzGetNodes
     \tkzDrawCircles(O,B)
     \tkzDrawCircles[cyan](P,B)
     \tkzDrawCircles[red](w,E)
     \tkzDrawCircles[new](x,F)
     \tkzDrawSegments(A,G E,G C,G)
     \tkzDrawPolygons[new](A,E,C A',E',C')
     \tkzDrawPoints(A,...,G,A',E',C',O,P)
     \begin{scope}[font=\scriptsize]
     \tkzLabelPoints(A,...,F)
     \tkzLabelPoints[above left](G,A',E',C')
     \tkzLabelCircle[left](O,B)(30){$(\beta)$}
     \tkzLabelCircle[below](P,A)(40){$(\gamma)$}
     \tkzLabelCircle[right](w,C)(90){$(\alpha)$}
     \tkzLabelCircle[left](x,B)(-230){$((\delta))$}
     \end{scope}
  \end{tikzpicture}
```



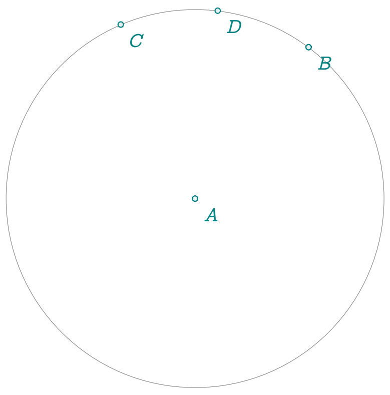
23.51 Three tangents

```
\begin{tkzelements}
   z.A    = point: new (-1 , 0)
   z.C    = point: new (4 , -1.5)
   z.E    = point: new (1 , -1)
   z.F    = point: new (1.5 , 2.5)
   T.AEF  = triangle : new (z.A,z.E,z.F)
   T.CEF  = triangle : new (z.C,z.E,z.F)
   z.w    = T.AEF.circumcenter
   z.x    = T.CEF.circumcenter
   C.wE   = circle : new (z.w,z.E)
   C.xE   = circle : new (z.x,z.E)
   L.Aw   = line : new (z.A,z.w)
   L.Cx   = line : new (z.C,z.x)
   z.G    = intersection (L.Aw,L.Cx)
   L.TA   = C.wE : tangent_at (z.A)
   L.TC   = C.xE : tangent_at (z.C)
   z.I    = intersection (L.TA,L.TC)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawCircles(w,E)
   \tkzDrawCircles[cyan](x,E)
   \tkzDrawCircles[red](G,A)
   \tkzDrawLines(A,I C,I F,I)
   \tkzDrawPoints(A,C,E,F)
   \tkzLabelPoints[right](A)
   \tkzLabelPoints[above right](E,F)
   \tkzLabelPoints[below](C)
\end{tikzpicture}
```

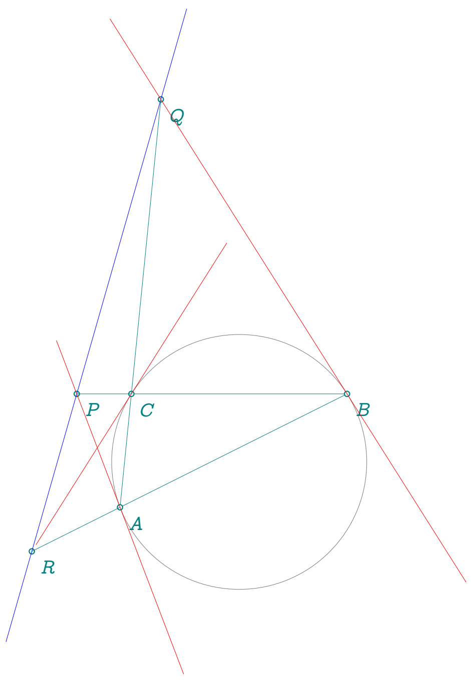


23.52 Midarc

```
\begin{tkzelements}
   z.A   = point:  new (-1,0)
   z.B   = point:  new (2,4)
   C.AB  = circle: new (z.A,z.B)
   z.C   =  z.A: rotation (math.pi/3,z.B)
   z.D   = C.AB: midarc (z.B,z.C)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPoints(A,B,C)
   \tkzDrawCircles(A,B)
   \tkzDrawPoints(A,...,D)
   \tkzLabelPoints(A,...,D)
\end{tikzpicture}
```

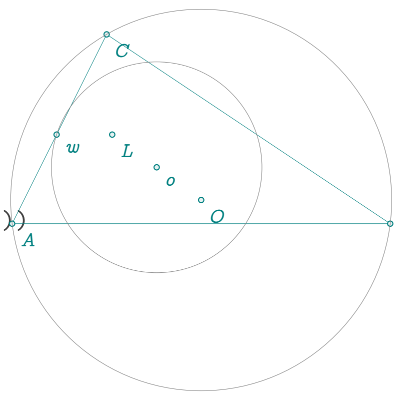


23.53 Lemoine Line without macro



```
\begin{tkzelements}
   scale        = 1.6
   z.A          = point: new (1,0)
   z.B          = point: new (5,2)
   z.C          = point: new (1.2,2)
   T            = triangle: new(z.A,z.B,z.C)
   z.O          = T.circumcenter
   L.AB         = line: new (z.A,z.B)
   L.AC         = line: new (z.A,z.C)
   L.BC         = line: new (z.B,z.C)
   C.OA         = circle: new (z.O,z.A)
   z.Ar,z.Al    = get_points (C.OA: tangent_at (z.A))
   z.Br,z.Bl    = get_points (C.OA: tangent_at (z.B))
   z.Cr,z.Cl    = get_points (C.OA: tangent_at (z.C))
   L.tA         = line: new (z.Ar,z.Al)
   L.tB         = line: new (z.Br,z.Bl)
   L.tC         = line: new (z.Cr,z.Cl)
   z.P          = intersection (L.tA,L.BC)
   z.Q          = intersection (L.tB,L.AC)
   z.R          = intersection (L.tC,L.AB)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygon[teal](A,B,C)
   \tkzDrawCircle(O,A)
   \tkzDrawPoints(A,B,C,P,Q,R)
   \tkzLabelPoints(A,B,C,P,Q,R)
   \tkzDrawLine[blue](Q,R)
   \tkzDrawLines[red](Ar,Al Br,Q Cr,Cl)
   \tkzDrawSegments(A,R C,P C,Q)
\end{tikzpicture}
```

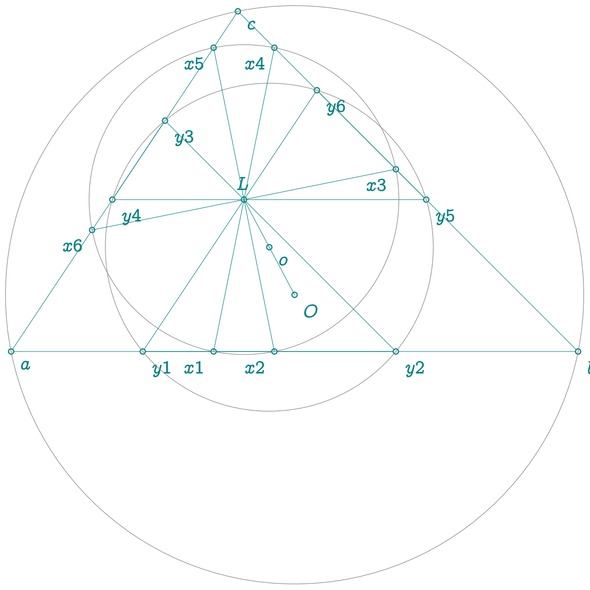
23.54 First Lemoine circle



```
\begin{tkzelements}
   z.A      = point:  new (1,1)
   z.B      = point:  new (5,1)
   z.C      = point:  new (2,3)
   T        = triangle: new (z.A,z.B,z.C)
   z.O      = T.circumcenter
   z.o,z.w  = get_points (T : first_lemoine_circle ())
   z.L      = T : lemoine_point ()
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygons(A,B,C)
   \tkzDrawPoints(A,B,C,o,w,O,L)
   \tkzLabelPoints(A,B,C,o,w,O,L)
   \tkzDrawCircles(o,w  O,A)
\end{tikzpicture}
```

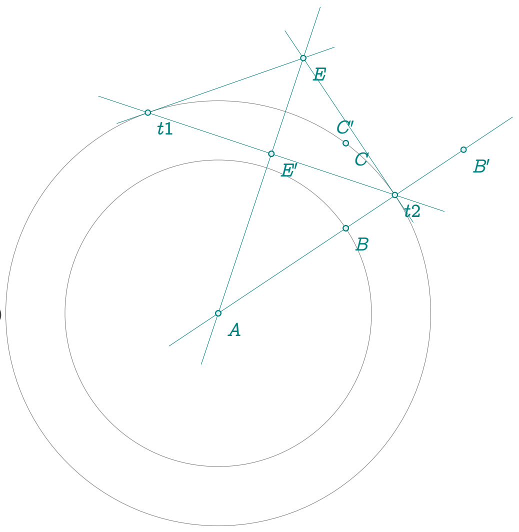
23.55 First and second Lemoine circles

```
\begin{tkzelements}
   scale              = 2
   z.a                = point:  new (0,0)
   z.b                = point:  new (5,0)
   z.c                = point:  new (2,3)
   T                  = triangle: new (z.a,z.b,z.c)
   z.O                = T.circumcenter
   z.o,z.p            = get_points (T : first_lemoine_circle ())
   L.ab               = line : new (z.a,z.b)
   L.ca               = line : new (z.c,z.a)
   L.bc               = line : new (z.b,z.c)
   z.L,z.x            = get_points (T : second_lemoine_circle ())
   C.first_lemoine    = circle : new (z.o,z.p)
   z.y1,z.y2          = intersection (L.ab,C.first_lemoine)
   z.y5,z.y6          = intersection (L.bc,C.first_lemoine)
   z.y3,z.y4          = intersection (L.ca,C.first_lemoine)
   C.second_lemoine   = circle : new (z.L,z.x)
   z.x1,z.x2          = intersection (L.ab,C.second_lemoine)
   z.x3,z.x4          = intersection (L.bc,C.second_lemoine)
   z.x5,z.x6          = intersection (L.ca,C.second_lemoine)
   L.y1y6             = line : new (z.y1,z.y6)
   L.y4y5             = line : new (z.y4,z.y5)
   L.y2y3             = line : new (z.y2,z.y3)
\end{tkzelements}
\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygons(a,b,c y1,y2,y3,y4,y5,y6)
   \tkzDrawPoints(x1,x2,x3,x4,x5,x6,L)
   \tkzDrawPoints(a,b,c,o,O,y1,y2,y3,y4,y5,y6)
   \tkzLabelPoints[below right](a,b,c,o,O,y1,y2,y3,y4,y5,y6)
   \tkzLabelPoints[below left](x1,x2,x3,x4,x5,x6)
   \tkzLabelPoints[above](L)
   \tkzDrawCircles(L,x o,p O,a)
   \tkzDrawSegments(L,O x1,x4 x2,x5 x3,x6)
\end{tikzpicture}
```



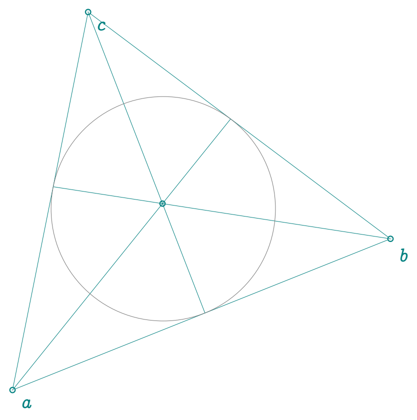
23.56 Inversion

```
\begin{tkzelements}
  z.A      = point: new (-1,0)
  z.B      = point: new (2,2)
  z.C      = point: new (2,4)
  z.E      = point: new (1,6)
  C.AC     = circle:    new (z.A,z.C)
  L.Tt1,
  L.Tt2    = C.AC: tangent_from (z.E)
  z.t1     = L.Tt1.pb
  z.t2     = L.Tt2.pb
  L.AE     = line: new (z.A,z.E)
  z.H      = L.AE : projection (z.t1)
  z.Bp,
  z.Ep,
  z.Cp     = C.AC: inversion ( z.B, z.E, z.C )
\end{tkzelements}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawPoints(A,B,C)
  \tkzDrawCircles(A,C A,B)
  \tkzDrawLines(A,B' E,t1 E,t2 t1,t2 A,E)
  \tkzDrawPoints(A,B,C,E,t1,t2,H,B',E')
  \tkzLabelPoints(A,B,C,E,t1,t2,B',E')
  \tkzLabelPoints[above](C')
\end{tikzpicture}
```



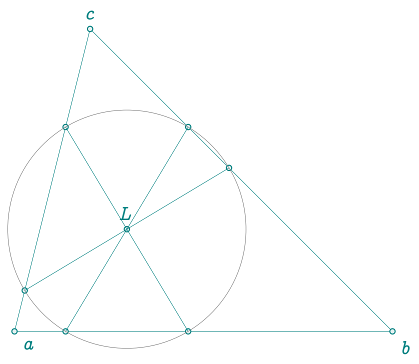
23.57 Gergonne point

```
\begin{tkzelements}
z.a  = point: new(1,0)
z.b  = point: new(6,2)
z.c  = point: new(2,5)
T    = triangle : new (z.a,z.b,z.c)
z.g  = T : gergonne_point ()
z.i  = T.incenter
z.ta,z.tb,z.tc = get_points (T :  intouch ())
\end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygons(a,b,c)
\tkzDrawPoints(a,b,c,g)
\tkzLabelPoints(a,b,c)
\tkzDrawSegments (a,ta b,tb c,tc)
\tkzDrawCircle(i,ta)
\end{tikzpicture}
```



23.58 Antiparallel through Lemoine point

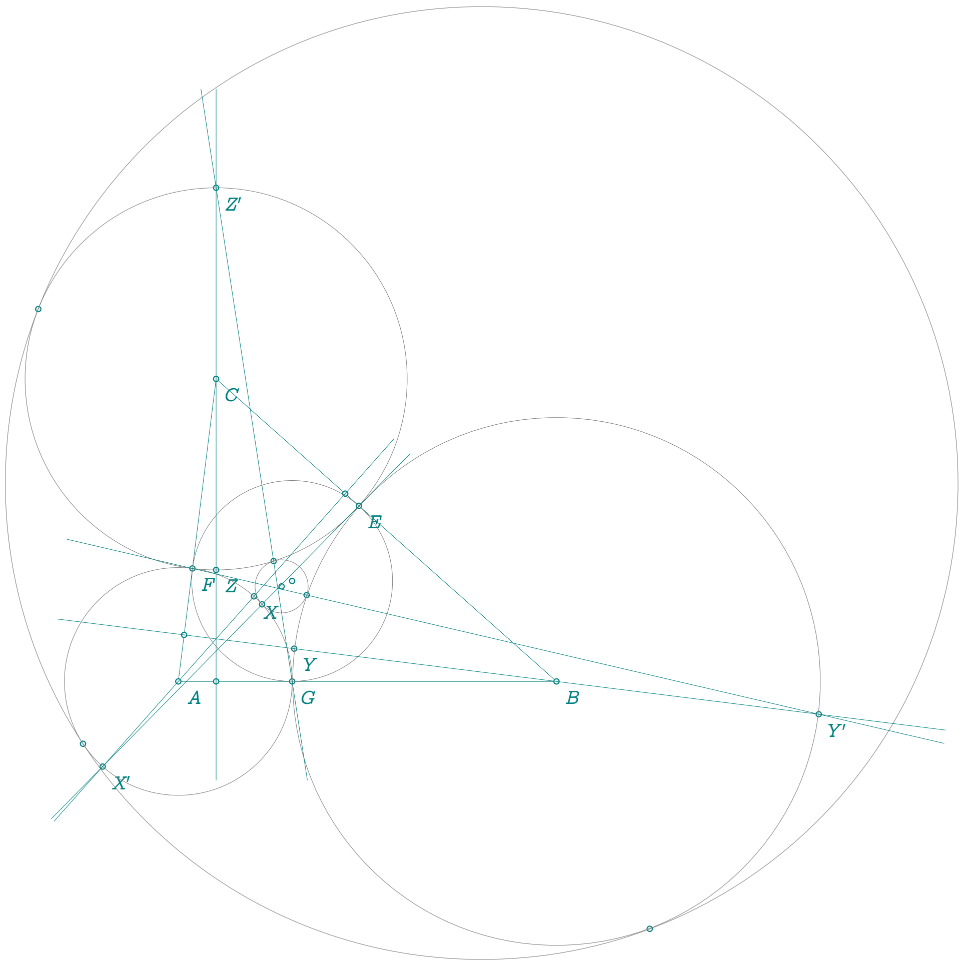
```
\begin{tkzelements}
   z.a         = point:  new (0,0)
   z.b         = point:  new (5,0)
   z.c         = point:  new (1,4)
   T           = triangle: new (z.a,z.b,z.c)
   z.L         = T : lemoine_point ()
   L.anti      = T : antiparallel (z.L,0)
   z.x_0,z.x_1 = get_points (L.anti)
   L.anti      = T : antiparallel (z.L,1)
   z.y_0,z.y_1 = get_points (L.anti)
   L.anti      = T : antiparallel (z.L,2)
   z.z_0,z.z_1 = get_points (L.anti)
\end{tkzelements}

\begin{tikzpicture}
   \tkzGetNodes
   \tkzDrawPolygons(a,b,c)
   \tkzDrawPoints(a,b,c,L,x_0,x_1,y_0,y_1,z_0,z_1)
   \tkzLabelPoints(a,b)
   \tkzLabelPoints[above](L,c)
   \tkzDrawSegments(x_0,x_1 y_0,y_1 z_0,z_1)
   \tkzDrawCircle(L,x_0)
\end{tikzpicture}
```



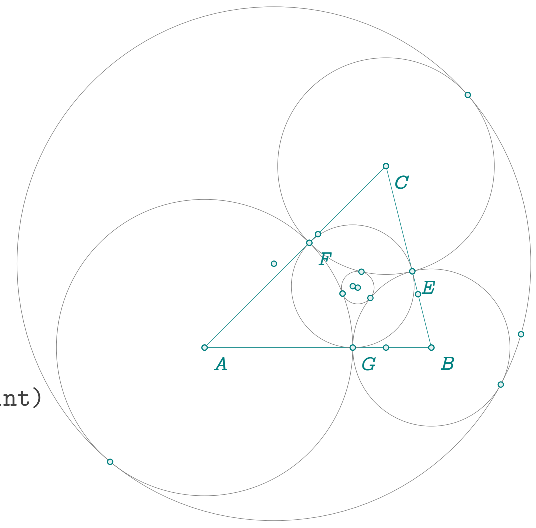
23.59 Soddy circle without function

```
\begin{tkzelements}
z.A = point : new ( 0   , 0   )
z.B = point : new ( 5   , 0   )
z.C = point : new ( 0.5 ,  4   )
T.ABC = triangle : new ( z.A,z.B,z.C )
z.I = T.ABC.incenter
```

```
z.E,z.F,z.G = T.ABC : projection (z.I)
C.ins = circle : new (z.I,z.E)
T.orthic = T.ABC : orthic ()
z.Ha,z.Hb,z.Hc = get_points (T.orthic)
C.CF = circle : new ( z.C , z.F )
C.AG = circle : new ( z.A , z.G )
C.BE = circle : new ( z.B , z.E )
L.Ah = line : new ( z.A , z.Ha )
L.Bh = line : new ( z.B , z.Hb )
L.Ch = line : new ( z.C , z.Hc )
z.X,z.Xp = intersection (L.Ah,C.AG)
z.Y,z.Yp = intersection (L.Bh,C.BE)
z.Z,z.Zp = intersection (L.Ch,C.CF)
L.XpE = line   : new (z.Xp,z.E)
L.YpF = line   : new (z.Yp,z.F)
L.ZpG = line   : new (z.Zp,z.G)
z.S = intersection (L.XpE,L.YpF)
z.Xi = intersection(L.XpE,C.AG)
z.Yi = intersection(L.YpF,C.BE)
_,z.Zi = intersection(L.ZpG,C.CF)
z.S = triangle : new (z.Xi,z.Yi,z.Zi).circumcenter
C.soddy_int = circle : new (z.S,z.Xi)
C.soddy_ext = C.ins : inversion (C.soddy_int)
z.w = C.soddy_ext.center
z.s = C.soddy_ext.through
z.Xip,z.Yip,z.Zip = C.ins : inversion (z.Xi,z.Yi,z.Zi)
\end{tkzelements}

\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C)
\tkzDrawPoints(A,B,C,E,F,G,Ha,Hb,Hc,X,Y,Z,X',Y',Z',Xi,Yi,Zi,I)
\tkzDrawPoints(Xi',Yi',Zi',S)
\tkzLabelPoints(A,B,C,E,F,G,X,Y,Z,X',Y',Z')
\tkzDrawCircles(A,G B,E C,F I,E S,Xi w,s)
\tkzDrawLines(X',Ha Y',Hb Z',Hc)
\tkzDrawLines(X',E Y',F Z',G)
\end{tikzpicture}
```



23.60 Soddy circle with function



```
  \begin{tkzelements}
  z.A = point : new ( 0   , 0   )
  z.B = point : new ( 5   , 0   )
  z.C = point : new (4 ,   4   )
  T.ABC = triangle : new ( z.A,z.B,z.C )
  z.I = T.ABC.incenter
  z.E,z.F,z.G = T.ABC : projection (z.I)
  T.orthic = T.ABC : orthic ()
  z.Ha,z.Hb,z.Hc = get_points (T.orthic)
  C.ins = circle : new (z.I,z.E)
  z.s,z.xi,z.yi,
  z.zi = T.ABC : soddy_center ()
  C.soddy_int = circle : new (z.s,z.xi)
  C.soddy_ext = C.ins : inversion (C.soddy_int)
  z.w = C.soddy_ext.center
  z.t = C.soddy_ext.through
  z.Xip,z.Yip,
  z.Zip = C.ins : inversion (z.xi,z.yi,z.zi)
  \end{tkzelements}
\begin{tikzpicture}
\tkzGetNodes
\tkzDrawPolygon(A,B,C)
```

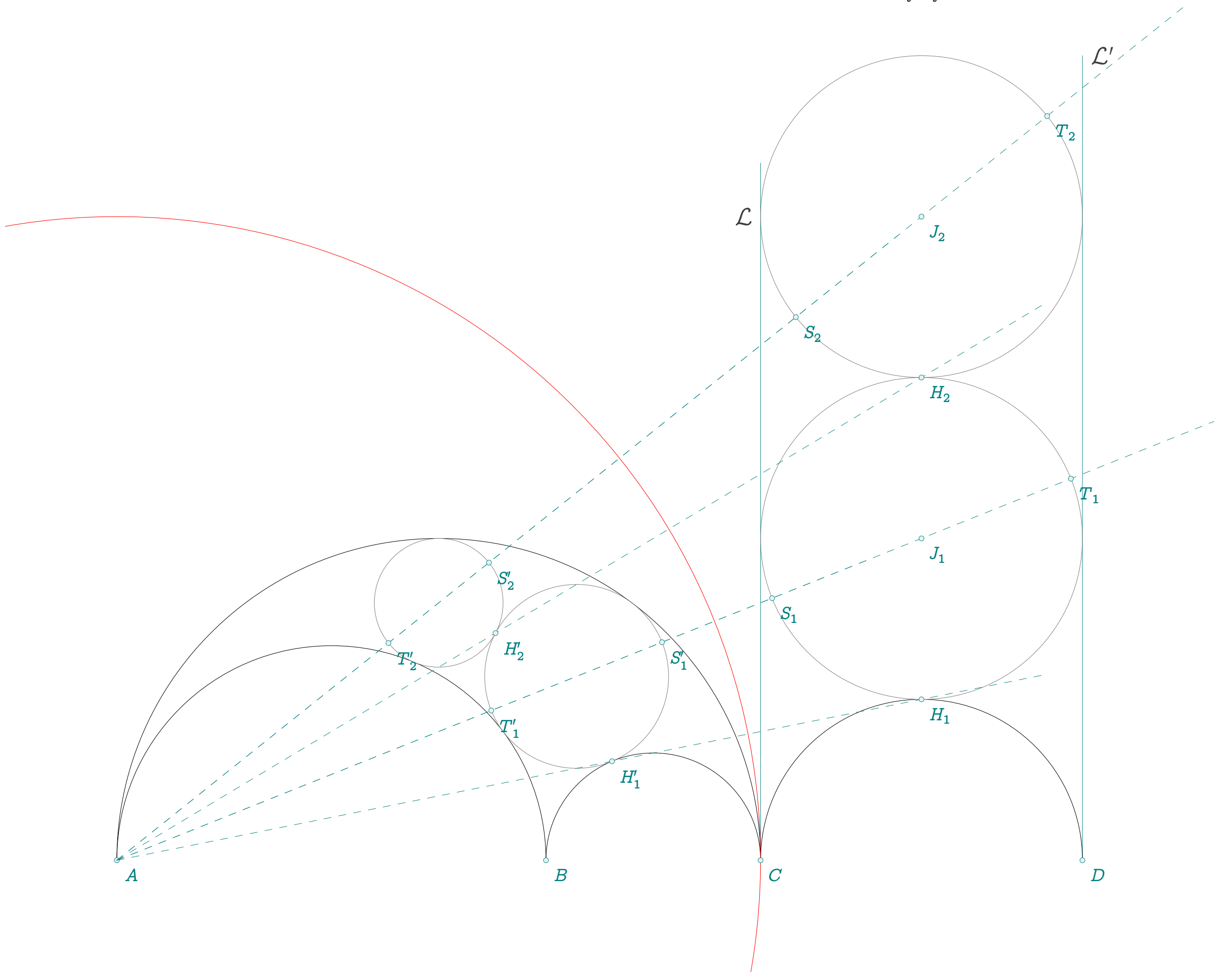
```
\tkzDrawCircles(A,G B,E C,F I,E s,xi w,t)
\tkzDrawPoints(A,B,C,E,F,G,s,w,xi,t)
\tkzLabelPoints(A,B,C)
\tkzDrawPoints(A,B,C,E,F,G,Ha,Hb,Hc,xi,yi,zi,I)
\tkzDrawPoints(Xi',Yi',Zi')
\tkzLabelPoints(A,B,C,E,F,G)
\end{tikzpicture}
```

23.60.1 Pappus chain

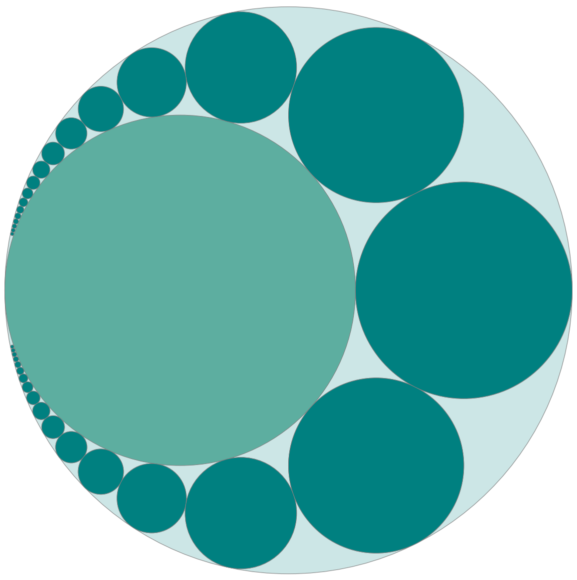
Soit le point D appartenant à la droite (AC) tel que

$$DB \cdot DA = AC^2$$

alors B est l'image de D dans l'inversion de centre A et puissance AC^2. Les demi-cercles de diamètre $[AB]$ et$[AC]$ passent par le pôle A. Ils ont pour images les demi-droites $\mathcal{L}'$ et $\mathcal{L}$.
Les cercles de centre J_i et de diamètre S_iT_i ont pour images les cercles de diamètre $S_i'T_i'$.

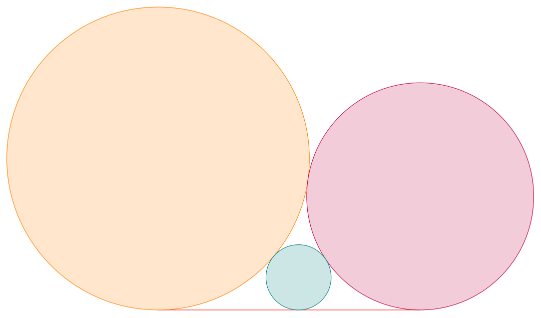


```
\begin{tkzelements}
  xC,nc     = 10,16
  xB        = xC/tkzphi
  xD        = (xC*xC)/xB
  xJ        = (xC+xD)/2
  r         = xD-xJ
  z.A       = point : new ( 0 , 0 )
  z.B       = point : new ( xB , 0)
  z.C       = point : new ( xC , 0)
  L.AC      = line : new (z.A,z.C)
  z.i       = L.AC.mid
  L.AB      = line:new (z.A,z.B)
  z.j       = L.AB.mid
  z.D       = point : new ( xD , 0)
  C.AC      = circle: new (z.A,z.C)
  for i     = -nc,nc do
     z["J"..i]    = point: new (xJ,2*r*i)
     z["H"..i]    = point: new (xJ,2*r*i-r)
     z["J"..i.."p"], z["H"..i.."p"]    = C.AC : inversion (z["J"..i],z["H"..i])
     L.AJ         = line : new (z.A,z["J"..i])
     C.JH         = circle: new ( z["J"..i] ,  z["H"..i])
     z["S"..i], z["T"..i]              = intersection (L.AJ,C.JH)
     z["S"..i.."p"], z["T"..i.."p"]    = C.AC : inversion (z["S"..i],z["T"..i])
     L.SpTp       = line:new (  z["S"..i.."p"], z["T"..i.."p"])
     z["I"..i]    = L.SpTp.mid
   end
\end{tkzelements}
```



```
\def\nc{\tkzUseLua{nc}}
\begin{tikzpicture}[ultra thin]
   \tkzGetNodes
   \tkzDrawCircle[fill=teal!20](i,C)
   \tkzDrawCircle[fill=PineGreen!60](j,B)
   \foreach \i in {-\nc,...,0,...,\nc} {
   \tkzDrawCircle[fill=teal]({I\i},{S\i'})
  }
\end{tikzpicture}
```

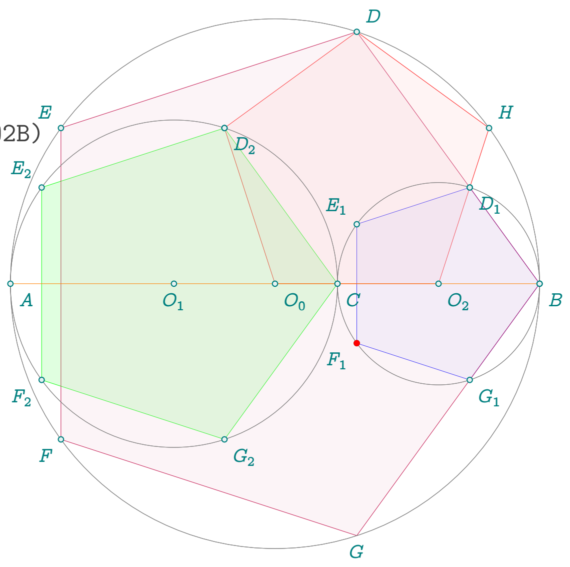
23.61 Three Circles



```
\begin{tkzelements}
function threecircles(c1,r1,c2,r2,c3,h1,h3,h2)
   local xk = math.sqrt (r1*r2)
   local cx = (2*r1*math.sqrt(r2))/(math.sqrt(r1)+math.sqrt(r2))
   local cy = (r1*r2)/(math.sqrt(r1)+math.sqrt(r2))^2
   z[c2] = point : new ( 2*xk , r2 )
   z[h2] = point : new (2*xk,0)
   z[c1] = point : new (0,r1)
   z[h1] = point : new (0,0)
   L.h1h2 = line: new(z[h1],z[h2])
   z[c3] = point : new (cx,cy)
   z[h3] = L.h1h2: projection (z[c3])
end
   threecircles("A",4,"B",3,"C","E","G","F")
\end{tkzelements}

\begin{tikzpicture}
  \tkzGetNodes
  \tkzDrawSegment[color = red](E,F)
  \tkzDrawCircle[orange,fill=orange!20](A,E)
  \tkzDrawCircle[purple,fill=purple!20](B,F)
  \tkzDrawCircle[teal,fill=teal!20](C,G)
\end{tikzpicture}
```

23.62 pentagons in a golden arbelos

```
\begin{tkzelements}
  z.A       = point: new (0 , 0)
  z.B       = point: new (10 , 0)
  L.AB      = line:  new ( z.A, z.B)
  z.C       = L.AB : gold_ratio ()
  L.AC      = line:  new ( z.A, z.C)
  L.CB      = line:  new ( z.C, z.B)
  z.O_0     = L.AB.mid
  z.O_1     = L.AC.mid
  z.O_2     = L.CB.mid
  C.O0B     = circle: new ( z.O_0, z.B)
  C.O1C     = circle: new ( z.O_1, z.C)
  C.O2B     = circle: new ( z.O_2, z.B)
  z.M_0     = C.O1C : external_similitude (C.O2B)
  L.O0C     = line:new(z.O_0,z.C)
  T.golden = L.O0C : golden ()
  z.L       = T.golden.pc
  L.O0L     = line:new(z.O_0,z.L)
  z.D       = intersection (L.O0L,C.O0B)
  L.DB      = line:new(z.D,z.B)
  z.Z       = intersection (L.DB,C.O2B)
  L.DA      = line:new(z.D,z.A)
  z.I       = intersection (L.DA,C.O1C)
  L.O2Z     = line:new(z.O_2,z.Z)
  z.H       = intersection (L.O2Z,C.O0B)
  C.BD      = circle:new (z.B,z.D)
  C.DB      = circle:new (z.D,z.B)
  _,z.G     = intersection (C.BD,C.O0B)
  z.E       = intersection (C.DB,C.O0B)
  C.GB      = circle:new (z.G,z.B)
  _,z.F     = intersection (C.GB,C.O0B)
  k         = 1/tkzphi^2
  kk        = tkzphi
  z.D_1,z.E_1,z.F_1,z.G_1 = z.B :   homothety (k, z.D,z.E,z.F,z.G)
  z.D_2,z.E_2,z.F_2,z.G_2 = z.M_0 : homothety (kk,z.D_1,z.E_1,z.F_1,z.G_1)
\end{tkzelements}
```



```
\begin{tikzpicture}[scale=.8]
\tkzGetNodes
\tkzDrawPolygon[red](O_2,O_0,I,D,H)
\tkzDrawPolygon[blue](B,D_1,E_1,F_1,G_1)
\tkzDrawPolygon[green](C,D_2,E_2,F_2,G_2)
\tkzDrawPolygon[purple](B,D,E,F,G)
\tkzDrawCircles(O_0,B O_1,C O_2,B)
\tkzFillPolygon[fill=red!20,opacity=.20](O_2,O_0,I,D,H)
\tkzFillPolygon[fill=blue!20,opacity=.20](B,D_1,E_1,F_1,G_1)
\tkzFillPolygon[fill=green!60,opacity=.20](C,D_2,E_2,F_2,G_2)
\tkzFillPolygon[fill=purple!20,opacity=.20](B,D,E,F,G)
\tkzDrawCircles(O_0,B O_1,C O_2,B)
\tkzDrawSegments[new](A,B)
\tkzDrawPoints(A,B,C,O_0,O_1,O_2,Z,I,H,B,D,E,F)
\tkzDrawPoints(D_1,E_1,F_1,G_1)
```

```
\tkzDrawPoints(D_2,E_2,F_2,G_2)
\tkzDrawPoints[red](F_1)
\tkzLabelPoints(A,B,C,O_0,O_2)
\tkzLabelPoints[below](O_1,G)
\tkzLabelPoints[above right](D,H)
\tkzLabelPoints[above left](E,E_1,E_2)
\tkzLabelPoints[below left](F,F_1,F_2)
\tkzLabelPoints(D_1,G_1)
\tkzLabelPoints(D_2,G_2)
\end{tikzpicture}
```

Index

24 Cheat_sheet

`r` denotes a real number, `cx` complex number, `d` a positive real number, `n` an integer, `an` an angle, `b` a boolean, `s` a character string, `pt` a point, `t` a table, `m` a matrix, `v` variable, `L` a straight line, `C` a circle, `T` a triangle, `E` an ellipse, `V` a vector,`Q` a quadrilateral, `P` a parallelogram, `R` a rectangle, `S` a square, `RP` a regular polygon, `M` a matrix, `O` an object (pt, L,C,T), `..` a list of points or an object, < > optional argument.

point

Attributes table(1)

`re`	`-> r`
`im`	`-> r`
`type`	`-> s`
`argument`	`-> r`
`modulus`	`-> d`

Functions table(1)

`new`	`-> pt`
`polar`	`-> pt`
`polar_deg`	`-> pt`

Methods table(29)

`+ - * /`	`(pt,pt)`	`-> pt`
`..`	`(pt,pt)`	`-> r`
`^`	`(pt,pt)`	`-> r`
`=`		`-> b`
`tostring`		`-> s`

Methods table(2) table(30)

`conj`	`-> pt`
`abs`	`-> r`
`mod`	`-> d`
`norm`	`-> d`
`arg`	`-> d`
`get`	`-> r,r`
`sqrt`	`-> pt`
`north(d)`	`-> pt`
`south(d)`	`-> pt`
`east(d)`	`-> pt`
`west(d)`	`-> pt`
`normalize(pt)`	`-> pt`
`symmetry (...)`	`-> O`
`rotation (an , ...)`	`-> O`
`homothety (r , ...)`	`-> O`
`orthogonal(d)`	`-> pt`
`at()`	`-> pt`
`print()`	`-> s`

line

Attributes table(3)

`pa,pb`	`-> pt`
`type`	`-> s`
`mid`	`-> pt`
`north_pa`	`-> pt`
`north_pb`	`-> pt`
`south_pa`	`-> pt`
`south_pb`	`-> pt`
`east`	`-> pt`
`west`	`-> pt`
`slope`	`-> r`
`length`	`-> d`
`vec`	`-> V`

Methods table(6)

`new (pt,pt)`	`-> d`
`distance (pt)`	`-> d`
`slope ()`	`-> r`
`in_out (pt)`	`-> b`
`in_out_segment (pt)`	`-> b`
`barycenter (r,r)`	`-> pt`
`point (t)`	`-> pt`
`midpoint ()`	`-> pt`
`harmonic_int (pt)`	`-> pt`
`harmonic_ext (pt)`	`-> pt`
`harmonic_both (d)`	`-> pt`
`gold_ratio()`	`-> pt`
`normalize ()`	`-> pt`
`normalize_inv ()`	`-> pt`
`_north_pa (d)`	`-> pt`
`_north_pb (d)`	`-> pt`
`_south_pa (d)`	`-> pt`
`_south_pb (d)`	`-> pt`
`_east (d)`	`-> pt`
`_west (d)`	`-> pt`
`translation (...)`	`-> O`
`projection (...)`	`-> O`
`reflection (...)`	`-> O`
`ll_from ( pt )`	`-> L`
`ortho_from ( pt )`	`-> L`
`mediator ()`	`-> L`
`circle ()`	`-> C`
`circle_swap ()`	`-> C`
`diameter ()`	`-> C`
`apollonius (r)`	`-> C`
`equilateral (<swap>)`	`-> T`
`isosceles (an,<swap>)`	`-> T`
`school ()`	`-> T`
`two_angles (an,an)`	`-> T`
`half ()`	`-> T`
`sss (r,r,r)`	`-> T`
`sas (r,an)`	`-> T`
`ssa (r,an)`	`-> T`
`gold (<swap>)`	`-> T`
`euclide (<swap>)`	`-> T`
`golden (<swap>)`	`-> T`
`divine ()`	`-> T`
`cheops ()`	`-> T`
`pythagoras ()`	`-> T`
`sublime ()`	`-> T`
`egyptian ()`	`-> T`
`square (<swap>)`	`-> T`
`report (r,pt)`	`-> T`

triangle

Attributes table(9)

`pa,pb,pc`	`-> pt`
`circumcenter`	`-> pt`
`centroid`	`-> pt`
`incenter`	`-> pt`
`eulercenter`	`-> pt`
`orthocenter`	`-> pt`
`spiekercenter`	`-> pt`
`type`	`-> s`
`a`	`-> d`
`b`	`-> d`
`c`	`-> d`
`ab`	`-> L`
`bc`	`-> L`
`ca`	`-> L`
`alpha`	`-> r`
`beta`	`-> r`
`gamma`	`-> r`

Methods table(10)

`new (pt,pt,pt)`	`-> pt`
`trilinear (r,r,r)`	`-> pt`
`barycentric (r,r,r)`	`-> pt`
`bevan_point ()`	`-> pt`
`mittenpunkt_point ()`	`-> pt`
`gergonne_point ()`	`-> pt`
`nagel_point ()`	`-> pt`
`feuerbach_point ()`	`-> pt`
`lemoine_point()`	`-> pt`
`symmedian_point()`	`-> pt`
`spieker_center()`	`-> pt`
`barycenter (r,r,r)`	`-> pt`
`base (u,v)`	`-> pt`
`euler_points ()`	`-> pt`
`nine_points ()`	`-> pt`
`point (t)`	`-> pt`
`soddy_center ()`	`-> pt`
`euler_line ()`	`-> L`
`symmedian_line (n)`	`-> L`
`altitude (n)`	`-> L`
`bisector (n)`	`-> L`
`bisector_ext(n)`	`-> L`
`antiparallel(pt,n)`	`-> L`
`euler_circle ()`	`-> C`
`circum_circle()`	`-> C`
`in_circle ()`	`-> C`
`ex_circle (n)`	`-> C`
`first_lemoine_circle()`	`-> C`
`second_lemoine_circle()`	`-> C`
`spieker_circle()`	`-> C`
`soddy_circle ()`	`-> C`
`orthic()`	`-> T`

`medial()`	-> T
`incentral()`	-> T
`excentral()`	-> T
`intouch()`	-> T
`contact()`	-> T
`extouch()`	-> T
`feuerbach()`	-> T
`anti ()`	-> T
`tangential ()`	-> T
`cevian (pt)`	-> T
`symmedian ()`	-> T
`euler ()`	-> T
`projection (pt)`	-> pt,pt,pt
`parallelogram ()`	-> pt
`area ()`	-> d
`barycentric_coordinates(pt)`	-> r,r,r
`in_out (pt)`	-> pt
`check_equilateral ()`	-> b

circle

Attributes table(7)

`center`	-> pt
`through`	-> pt
`north`	-> pt
`south`	-> pt
`east`	-> pt
`west`	-> pt
`opp`	-> pt
`type`	-> s
`radius`	-> d
`ct`	-> L

Methods table(8)

`new (pt,pt)`	-> C
`radius (pt, r)`	-> C
`diameter (pt,pt)`	-> C
`in_out (pt)`	-> b
`in_out_disk (pt)`	-> b
`circles_position (C)`	-> s
`power (pt)`	-> r
`antipode (pt)`	-> pt
`midarc (pt,pt)`	-> pt
`point (r)`	-> pt
`random_pt (lower, upper)`	-> pt
`internal_similitude (C)`	-> pt
`external_similitude (C)`	-> pt
`radical_center(C,<C>)`	-> pt
`tangent_at (pt)`	-> L
`radical_axis (C)`	-> L
`radical_circle(C,<C>)`	-> C
`orthogonal_from (pt)`	-> C
`orthogonal_through(pt,pt)`	-> C
`midcircle(C)`	-> C
`external_tangent(C)`	-> L,L
`internal_tangent(C)`	-> L,L
`common_tangent(C)`	-> L,L
`tangent_from (pt)`	-> L,L
`inversion (...)`	-> O

ellipse

Attributes table(12)

`center`	-> pt
`vertex`	-> pt
`covertex`	-> pt
`Fa`	-> pt
`Fb`	-> pt
`north`	-> pt
`south`	-> pt
`east`	-> pt
`west`	-> pt
`Rx`	-> d
`Ry`	-> d
`slope`	-> r
`type`	-> s

Methods table(12)

`new (pt,pt,pt)`	-> E
`foci (pt,pt,pt)`	-> E
`radii (pt,r,r,an)`	-> E
`in_out (pt)`	-> b
`tangent_at (pt)`	-> L
`tangent_from (pt)`	-> L
`point (r)`	-> pt

square

Attributes table(15)

`pa,pb,pc,pd`	-> pt
`type`	-> s
`side`	-> d
`center`	-> pt
`exradius`	-> d
`inradius`	-> d
`diagonal`	-> d
`proj`	-> pt
`ab bc cd da`	-> L
`ac bd`	-> L

Methods table(16)

`new (pt,pt,pt,pt)`	-> S
`rotation (pt,pt)`	-> S
`side (pt,pt,<swap>)`	-> S

rectangle

Attributes table(17)

`pa,pb,pc,pd`	-> pt
`type`	-> s
`center`	-> pt
`exradius`	-> d
`length`	-> r
`width`	-> r
`diagonal`	-> d
`ab bc cd da`	-> L
`ac bd`	-> L

Methods table(18)

`new (pt,pt,pt,pt)`	-> R
`angle (pt,pt,an)`	-> R
`gold (pt,pt,<swap>)`	-> R
`diagonal (pt,pt,<swap>)`	-> R
`side (pt,pt,r,<swap>)`	-> R
`get_lengths ()`	->r,r

quadrilateral

Attributes table(13)

`pa,pb,pc,pd`	-> pt
`ab bc cd da`	-> L
`ac bd`	-> L
`type`	-> s
`i`	-> pt
`g`	-> pt
`a b c d`	-> r

Methods table(14)

`new (pt,pt,pt,pt)`	-> Q
`iscyclic ()`	-> b

parallelogram

Attributes table(19)

`pa,pb,pc,pd`	-> pt
`ab bc cd da`	-> L
`ac bd`	-> L
`type`	-> s
`center`	-> pt

Methods table(20)

`new (pt,pt,pt,pt)`	->
`fourth (pt,pt,pt)`	->

Regular_polygon

Attributes table(21)

`center`	-> pt
`through`	-> pt
`circle`	-> C
`type`	-> s
`side`	-> d
`exradius`	-> d
`inradius`	-> d
`proj`	-> pt
`nb`	-> i
`angle`	-> an

Methods table(22)

`new (pt,pt,n)`	-> PR
`incircle ()`	-> C
`name (s)`	-> ?

vector

Attributes table(23)

`type`	-> s
`norm`	-> d
`slope`	-> r
`mtx`	-> M

Methods table(24)

`new (pt,pt)`	-> V
`+ - *`	-> pt
`normalize (V)`	-> V
`orthogonal (d)`	-> V
`scale (r)`	-> V

`at (pt)`	`-> V`

matrix

Attributes table(25)

`set`	`-> t`
`rows`	`-> n`
`cols`	`-> n`
`type`	`-> s`
`det`	`-> r`

Functions table(27)

`new`	`-> m`
`square`	`-> m`
`htm`	`-> m`
`vector`	`-> m`

Metamethods table(26)

`+ - * (m,m)`	`-> m`
`^ (m,n)`	`-> m`
`=`	`-> b`
`tostring`	`-> s`

Method table(27)

`print`	`-> s`
`get`	`-> r/cx`
`inverse`	`-> m`
`adjugate`	`-> m`
`transpose`	`-> m`
`is_diagonal`	`-> b`
`is_orthogonal`	`-> b`
`homogenization`	`-> m`
`htm_apply`	`-> m`

Misc.

Attributes table(28)

`scale (default =1)`	`-> r`
`tkzphi`	`-> r`
`tkzinvphi`	`-> r`
`tkzsqrtphi`	`-> r`
`tkz_epsilon (default=1e-8)`	`-> r`
`length`	`-> d`
`islinear(pt,pt,pt)`	`-> b`
`isortho(pt,pt,pt)`	`-> b`
`value{r}`	`-> r`
`real`	`-> r`
`angle_normalize (an)`	`-> an`
`barycenter (...)`	`-> pt`
`bisector (pt,pt,pt)`	`-> L`
`bisector_ext (pt,pt,pt)`	`-> L`
`altitude (pt,pt,pt)`	`-> L`
`midpoint (pt,pt)`	`-> pt`
`equilateral (pt,pt)`	`-> T`
`format_number(r,n)`	`-> r`
`solve_quadratic(cx,cx,cx)`	`-> cx,cx`
`\tkzUseLua{v}`	`-> s`

Macros

`\tkzDN[n]{r}`	`-> r`
`\tkzDrawLuaEllipse((pt,pt,pt))`	