

Theorem 1 (Dominated convergence of Lebesgue) Assume that g is an integrable function defined on the measurable set E and that $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions so that $|f_n| \leq g$. If f is a function so that $f_n \rightarrow f$ almost everywhere then

$$\lim_{n \rightarrow \infty} \int f_n = \int f.$$

Proof: The function $g - f_n$ is non-negative and thus from Fatou lemma we have that $\int (g - f) \leq \liminf \int (g - f_n)$. Since $|f| \leq g$ and $|f_n| \leq g$ the functions f and f_n are integrable and we have

$$\int g - \int f \leq \int g - \limsup \int f_n,$$

so

$$\int f \geq \limsup \int f_n.$$